GIRTH ALTERNATIVE FOR HNN EXTENSIONS

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ABSTRACT. The girth of a graph is the length of a shortest cycle in it. If the graph does not contain any cycles (that is, it is a forest), its girth is defined to be infinity. For a finitely generated group, we can define its girth as the supremum of girths of Cayley graphs of it with respect to all finite generating sets. A given class of finitely generated groups is said to satisfy Girth Alternative if any group from this class is either virtually solvable or has infinite girth. We prove the Girth Alternative for a sub-class of HNN extensions as well as for a sub-class of amalgamated free products of finitely generated groups, and indicate counterexamples to show that beyond our class, the alternative fails in general. We also prove the Girth Alternative for HNN extensions of non-elementary word hyperbolic groups.

1. INTRODUCTION

Let $G = \langle S|R\rangle$ be a finitely generated group and let A and B be two subgroups of G with isomorphism $\phi : A \to B$. The *HNN extension group of G relative to subgroups A and B* with stable letter t, denoted by (G, A, B, t) is the extended group containing G defined as

$$G^*_\phi = (G, A, B, t) = \langle G, t|t^{-1}at = \phi(a) \text{ for } a \in A\rangle$$

where A and B are not only isomorphic but also conjugate via the map ϕ. The HNN extensions have been introduced in combinatorial group theory by Higman-Neumann-Neumann in [17]; the notion also arises in topology as the fundamental group of a topological space when two subspaces are glued along a homeomorphism. [1] In recent decades, HNN extensions have been used as a popular tool to construct examples and counterexamples of groups for questions in combinatorial group theory.

In [22], Schleimer defined the girth of G with respect to a finite generating set S, denoted by $girth(G, S)$ as the length of the shortest non-trivial relation in G with respect to generating set S; and the girth of the group G is defined as

$$girth(G) = \sup_{S \subset G}\{girth(G, S) | \langle S\rangle = G, |S| < \infty\}.$$

For a given group G, it is natural to ask whether the $girth(G)$ is finite or infinite. In [22], [1], it is shown that finitely generated groups satisfying a law, which are not isomorphic to $\mathbb{Z}$, have finite girth. Moreover, in [2], the author shows that for many classes of groups (word hyperbolic, one-relator, linear groups not isomorphic to $\mathbb{Z}$), the property of having infinite girth coincides with the property of containing a non-abelian free subgroup and introduced the notion of Girth Alternative similar in spirit to the well-known Tits Alternative. *For a given class $\mathcal{C}$ of finitely generated groups, $\mathcal{C}$ is said to satisfy the Girth Alternative if any group from the class $\mathcal{C}$ has either infinite girth or is virtually solvable.*

[1]In this case one obtains a somewhat more general notion where A and B are not necessarily subgroups of G but have homomorphic images in G.

In [2], [3], Akhmedov has proved the Girth Alternative for the class of hyperbolic, linear, one-relator and $PL_+(I)$ groups. In [24], Yamagata proves the Girth Alternative for convergence groups and irreducible subgroups of the mapping class groups. Independently in [21], Nakamura proves the Alternative for all subgroups of mapping class groups and also for the subgroups of $\mathrm{Out}(\mathbb{F}_n)$ containing the irreducible elements having irreducible powers.

The table below shows a dichotomy between the Girth Alternative and Tits Alternative for some classes of finitely generated groups that we are interested in.

Groups	**Tits Alternative**	**Girth Alternative**
$\mathrm{PL}_+(\mathrm{I})$	fails (Thompson's group F)	holds
Linear	holds	holds
1-relator	holds	holds
Hyperbolic	holds	holds
$\mathrm{Homeo}_+(I)$	fails (Thompson's group F)	fails (we prove in [5])
$\mathrm{Diff}_+^\omega(I)$	**unknown**	holds (we prove in [4])
Residually finite	fails	fails
Group of formal power series (over field $\mathbf{k}$)	fails for char$\mathbf{k} > 0$ & **unknown** for char$\mathbf{k} = 0$	**fails** for char$\mathbf{k} > 0$ & holds for char$\mathbf{k}$=0
HNN Extensions	holds for proper extensions	holds for proper extensions (we prove in this paper)

In [1], an example (due to A.Olshanskii) of a group is mentioned which contains a copy of $\mathbb{F}_2$ but has finite girth; so, there are classes of groups where Tits Alternative holds but Girth Alternative fails. In [15], the authors construct an example of a finitely generated residually p-group which is not virtually solvable, but satisfies a law. This example shows that Tits Alternative and Girth Alternative fail in the class of residually finite groups. Moreover, since every countably based pro-p group embeds into the group of formal analytic power series over $\mathbb{F}_p$ [14], one can deduce that both alternatives fail in the latter group as well.

In this paper, our main result is the Girth Alternative for a sub-class of HNN extensions, showing for these sub-classes, again the property of having infinite girth coincide with the property of containing a non-abelian free subgroup.

1.1. **Conventions.** For a given group G, we say the HNN extension (G, A, B, t) is

1. *Proper*, when both underlying subgroups A and B are proper in G.
2. *Semi-proper*, when one of the subgroups is proper and the other is the full group G.
3. *Full*, when both A and B are full group G. [2]

Theorem 1.1. *For G be a finitely generated group with A and B two proper subgroups then $girth(\Gamma) = \infty$, where $\Gamma = (G, A, B, t)$ is a proper HNN extension of G relative to A, B and ϕ.*

Since proper HNN extensions are never solvable (they contain a subgroup isomorphic to $\mathbb{F}_2$), Theorem 1.1 implies Girth Alternative for proper HNN extensions. However, the following result provides a class of counterexamples to show that beyond our sub-class as in Theorem 1.1, the alternative fails in general.

[2]our terminology may differ from the terminology of many other authors; for example, a semi-proper HNN extension is often called (e.g. in [9], [10]) an ascending (or strictly ascending) HNN extension in many sources; in [10], "proper" is used for HNN extensions which are either proper or semiproper in our terminology.

Proposition 1.2. *For G a finitely generated group satisfying a law with $A = G$ and B a proper subgroup of G, then girth$(\Gamma) < \infty$, where $\Gamma = (G, A, B, t)$ is a semi proper HNN extension relative to G, B and ϕ.*

The proof of this proposition is easier; we simply observe that under the given hypothesis, $\Gamma = (G, A, B, t)$ will satisfy a law and will not be cyclic. In Section 4, we discuss a particulaly interesting case when G is a nilpotent group.

Treating the case of amalgamated free product of groups uses significantly different ideas; we provide a separate proposition devoted to this case.

Proposition 1.3. *Any proper amalgamated free product $A *_C B$, where A, B are finitely generated groups and $\max\{A : C, B : C\} \geq 3$, has infinite girth.*

We call an amalgamated free product proper if C is a proper subgroup in both A and B. Notice that without the condition $\max\{A : C, B : C\} \geq 3$ the claim does not hold since $girth(D_\infty) = 2 < \infty$. [3]

Interestingly, we also obtain the following result which shows that the Girth Alternative holds in general for the class of any HNN extension (proper, semi proper or full) of the non-abelian free group F_n for $n \geq 2$.

Proposition 1.4. *Any HNN extension of a non-elementary word hyperbolic group has infinite girth.*

As a special case, we obtain the following

Corollary 1.5. Any HNN extension of the non-abelian free group $\mathbb{F}_n$ for any $n \geq 2$ has infinite girth.

Interestingly, the claim of this corollary does not seem to lend itself to more elementary methods, or to known results in the literature about HNN extension of free groups. We discuss and emphasize some relevant problematic issues in Section 6. Notice that in the case of $n = 1$, Corollary 1.5 easily fails even for semi-proper HNN extensions since the Baumslag-Solitar group $BS(1, m) = \langle a, b \mid aba^{-1} = b^m \rangle$ is solvable hence has finite girth. On the other hand, proper HNN extensions of $\mathbb{Z}$ are all non-solvable one-relator groups, hence, by Theorem 3.1 in [2], have infinite girth. Let us also point out that HNN extensions of a free group $\mathbb{F}_k$ are not necessarily linear; for $k = 1$, recall that the groups $BS(n, m) = \langle a, b \mid ab^n a^{-1} = b^m \rangle$ are non-Hopfian hence not linear for all $n, m > 1, (m, n) = 1$, and for $k \geq 2$, examples are provided in [16]

2. Preliminary results

First, we prove the following proposition which seems interesting to us also from a purely combinatorial point of view.

Proposition 2.1. *Let G be a finitely generated group such that no quotient of G is isomorphic to a dihedral group $D_n, n \in (\mathbb{N}\backslash\{0, 1\}) \cup \{\infty\}$, and let A, B be proper isomorphic subgroups of G. Then G admits a finite generating set S such that $S \cap (A \cup B) = \emptyset$.*

[3] We use the notation D_∞ for the infinite dihedral group $\langle a, b \mid a^2 = b^2 = 1\rangle$. It is an infinite virtually cyclic group. Recall that a finite dihedral group $D_n, n \geq 2$ is given by the presentation $\langle a, b \mid a^2 = b^2 = (ab)^n = 1\rangle$. The term *dihedral group* will refer to D_q where $q \in (\mathbb{N}\backslash\{0, 1\}) \cup \{\infty]$.

Proof. Let $d = d(G)$ be the minimal cardinality of a generating set of G and

$$\mathcal{S} = \{S \subset G | |S| = d, \langle S \rangle = G\}.$$

For cyclic groups the claim is obvious (recalling that a subgroup of a cyclic group is cyclic) so we will assume that G is non-cyclic. Then we have $d \geq 2$. We introduce the following quantities:

$$\alpha(S) = |S \cap (A \backslash B)|, \beta(S) = |S \cap (B \backslash A)|$$
$$\gamma(S) = |S \cap (A \cap B)|, \delta(S) = |S \backslash (A \cup B)|$$

We now claim that there exists a finite generating set $S \in \mathcal{S}$ such that $\delta(S) = d$ (i.e. $\alpha(S) = 0, \beta(S) = 0, \gamma(S) = 0$). Indeed, let $S \in \mathcal{S}$ such that $\delta(S)$ is maximal. Assume $\delta(S) < d$.

Claim 1: $\alpha(S) = 0$ or $\beta(S) = 0$.

Proof: Indeed, assume that $\alpha(S) \geq 1$ and $\beta(S) \geq 1$, with $s_1 \in S \cap (A \backslash B)$ and $s_2 \in S \cap (B \backslash A)$. Then replace S with $S' = (S \backslash \{s_1\}) \cup \{s_1 s_2\}$. Since $s_1 s_2 \notin A \cup B$, we obtain that $\delta(S') = \delta(S) + 1$, contradicting maximality of $\delta(S)$.

Thus, without loss of generality, we may and will assume that $\beta(S) = 0$. Notice that $\delta(S) > 0$ because A is a proper subgroup of G.

Claim 2: $\alpha(S) + \gamma(S) \leq 1$

Proof. For S with $\beta(S) = 0$, suppose $\alpha(S) + \gamma(S) > 1$, and let $s_1, s_2 \in S \cap A, s_1 \neq s_2$. Let also $s_3 \in S \backslash (A \cup B)$, then $s_1 s_3 \notin A$ but by maximality of $\delta(S)$

$$s_1 s_3 \notin A \implies s_1 s_3 \in B.$$

Replace S by $S'' = \{s_1 s_3, s_2, s_3, \ldots, s_n\}$, note that as $\beta(S'') = 1 \neq 0$ then by Claim 1 we have $\alpha(S'') = 0$, which forces $s_2 \in S \cap (A \cap B)$. Similarly, let $S''' = \{s_2 s_3, s_1, s_3, \ldots, s_n\}$ and symmetrically, we obtain that $s_1 \in S \cap (A \cap B)$. Then $s_1 s_3 \notin B$. Contradiction.

Claim 3: $\alpha(S) + \gamma(S) = 0$, unless G has a quotient isomorphic to a quotient of D_∞.

Proof: Assuming the contrary, let $\alpha(S) + \gamma(S) = 1$ by Claim 2. Hence $|S \cap A| = 1$, so let $S \cap A = \{a\}$ and $S \backslash A = \{s_1, \ldots, s_{d-1}\}$. Notice that for all $1 \leq i \leq d-1$,

$$as_i \notin A, as_i \in B.$$ [4]

Choose $s_j \in S$ such that $s_j \notin B$ (such an s_j exists otherwise that would lead to $\delta(S) = 0$; but specifically, we have already assumed that $\delta(S)$ is maximal and $\beta(S) = 0$, so $s_j \notin B$ for all $1 \leq j \leq d-1$). Then, for all $1 \leq i \leq d-1$,

$$as_i s_j \in A, as_i s_j \notin B$$

So, using maximality of $\delta(S)$, inductively on the length of $m \geq 1$ of a reduced word $w = w(s_1, s_2, \ldots, s_{d-1})$ in the alphabet set $\{s_1^{\pm 1}, \ldots, s_{d-1}^{\pm 1}\}$, it follows that

$$aw \notin A, aw \in B \text{ if } m \text{ is odd},$$

$$aw \in A, aw \notin B \text{ if } m \text{ is even}.$$

Now, let $\mathcal{W}$ be the set of reduced words in the alphabet $\{s_1^{\pm 1}, \ldots, s_{d-1}^{\pm 1}\}$,

$$H = \{g \in G | g = w(s_1, \ldots, s_{d-1}) \text{ such that } w \in \mathcal{W}\},$$

[4]Since $a \in A, s_i \notin A$, clearly $as_i \notin A$ for all $1 \leq i \leq d-1$; and if $as_i \in B$ for some $1 \leq i \leq d-1$, then, since $\beta(S) = 0$, we can replace $S = \{a, s_1, \ldots, s_{d-1}\}$ with $S'''' = \{as_i, s_1, \ldots, s_{d-1}\}$ and obtain a generating set disjoint from $A \cup B$ with $\delta(S'''') = d$ hence we are done with the proof.

$H_1 = \{g \in H | g = w(s_1, \dots, s_{d-1})$ can be written as a word of even length in H in $s_1^{\pm 1}, \dots, s_{d-1}^{\pm 1}\}$.

Then, $[H : H_1] \leq 2$, hence $H_1 \trianglelefteq H$. Also, $aH_1 \subseteq A$, but note that for all $i \in \mathbb{Z}$,

$$aH_1 \subseteq A \implies H_1 \subseteq A \implies a^i H_1 a^{-i} \subseteq A. \tag{1}$$

Then, as $H_1 \trianglelefteq H$, we also have for all $1 \leq i \leq d-1$,

$$s_i H_1 s_i^{-1} \subseteq H_1 \subseteq A. \tag{2}$$

Any word in H_1 can be written as product of odd words $u^{-1}, v \in H$ and we know that $au, av \in B$ so we have,

$$u^{-1}v = u^{-1}a^{-1}av = (au)^{-1}av \in B \implies H_1 \subseteq B. \tag{3}$$

Similarly, $aH_1a^{-1} \subseteq B$ and for any u odd word in H

$$a^2 = auu^{-1}a = au(a^{-1}u)^{-1} \in B. \tag{4}$$

Finally, notice that

$$s_i a s_i^{-1} \in A \text{ for all } 1 \leq i \leq d-1 \tag{5}$$

because otherwise we can replace S with $S_1 = (S \backslash \{a\}) \cup \{s_i a s_i^{-1}\}$ and since $s_i a s_i^{-1} \notin B$ (because $a s_i^{-1} \in B$) we would have $S_1 \cap (A \cup B) = \emptyset$. From (1), (2), (3), (4) and (5) we get

$$N_G(H_1, a^2) \subseteq A \cap B.^5$$

Also, in the quotient $G/N_G(H_1, a^2)$ we have $\overline{s_i} = \overline{s_j}, 1 \leq i, j \leq d-1$ (i.e. the images of s_i and s_j are equal) thus by taking $b = s_1$ we obtain that this quotient is isomorphic to a quotient of infinite dihedral group,

$$D_\infty = \langle \bar{a}, \bar{b} | \bar{a}^2 = \bar{b}^2 = e \rangle$$

where $N_G(H_1, a^2)$ is the normal closure of $\{H_1, a^2\}$ in G. So, we get a quotient of G which is isomorphic to a quotient of D_∞ but any such quotient is isomorphic to $D_n, n \in (\mathbb{N} \backslash \{0, 1\}) \cup \{\infty\}$ which contradicts our assumption. □

From the proof, we immediately obtain the following proposition which will be of important use in the next section.

Proposition 2.2. *Let G be a finitely generated group, A, B be proper subgroups of G. Then there exists a finite generating set S of G with $|S \cap (A \cup B)| \leq 1$ and* $\min\{|S \cap (A \backslash B)|, |S \cap (B \backslash A)|\} = 0$.

Indeed, in the proof of Claim 3, we notice that $s_i \notin B, 1 \leq i \leq d-1$, because (since $as_i \in B$) otherwise $a \in B$ which contradicts our assumption there. □

Remark 2.3. For the group $D_\infty = \langle a, b \mid a^2 = b^2 = 1 \rangle$ with subgroups $K = \langle a, bab \rangle, L = \langle b, aba \rangle$ there exists no finite generating set S of G such that $S \cap (K \cup L) = \emptyset$. Indeed, $K \cup L$ contains all words of odd length of D_∞ hence no generating set of D_∞ is contained in $D_\infty \backslash (K \cup L)$. Also, the subgroups K and L are both maximal subgroups of D_∞.

[5]Here, $N_G(H_1, a^2)$ denotes the normal closure of the subset $H_1 \cup \{a^2\}$

It is useful to introduce the following property of finitely generated groups:

(P) We say a finitely generated group has property (P) if for all proper subgroups $A, B \leq G$, the group G admits a finite generating set S such that $S \subseteq G\backslash(A \cup B)$.

Let us note that the infinite dihedral group D_∞ has an obvious quotient isomorphic to Klein's Vierergruppe $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. On the other hand, if G surjects onto $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then by taking A, B as preimages of $(1,0)$ and $(0,1)$ respectively, we see hat there is no generating set S of G which is in $G\backslash(A \cup B)$ since the latter set is in the preimage of $(1,1)$. Thus, combined with this observation, Proposition 2.1 can be stated in the following cleaner form:

Proposition 2.4. *A finitely generated group has property (P) if and only if it does not surject onto the Klein's Vierergruppe.*

Remark 2.5. Proposition 2.4 (hence also Proposition 2.1, but not Proposition 2.2!) also follows from the classical result of Scorza ([6], [25]) which states that a group is a union of three proper subgroups if and only if it has a quotient isomorphic to Klein's Vierergruppe (we are grateful to the anonymous referee for pointing this out). Indeed, we just need to show the "if" direction, i.e. let G be a finitely generated group having no quotient isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and let A, B be proper subgroups. Let also S_0 be a finite generating set of G. By Scorza's Theorem, the subgroup of G generated by $G\backslash(A \cup B)$ is not proper (because this subgroup, together with A and B cover the entire G) hence the set $G\backslash(A \cup B)$ generates G. Thus $G\backslash(A \cup B)$ has a finite subset S which generates every element of S_0. Then S generates G and $S \subset G\backslash(A \cup B)$.

3. Proof of Theorem 1.1

First, we prove the following claim which shows the use of Proposition 2.1.

Proposition 3.1. *Let G be a group with a finite generating set S, and $G^*_\phi = (G, A, B, t)$ be an HNN extension where A, B are proper subgroups and $(A \cup B) \cap S = \emptyset$. Then $girth(G^*_\phi) = \infty$.*

Proof. The proof is a direct application of Britton's Lemma [8]. [6] Letting $S = \{s_1, \ldots, s_n\}$ where $1 \notin S$ for any $r \geq 2$, we can take $S^{(r)} = \{t, t^r s_1 t^{-2r}, t^{3r} s_2 t^{-4r}, \ldots, t^{(2n-1)r} s_n t^{-2nr}\}$. By Britton's Lemma, $girth(G^*_\phi, S^{(r)}) \geq r$ and since $r \geq 2$ is arbitrary, we conclude that $girth(G^*_\phi) = \infty$. □

We already know that we cannot always satisfy the condition of Proposition 3.1. The following proposition takes care of the situation not covered by Proposition 3.1.

Proposition 3.2. *Let G be a group with a finite generating set S, and $G^*_\phi = (G, A, B, t)$ be an HNN extension where A, B are proper subgroups and $|(A \cup B) \cap S| = 1$ and $\min\{|S \cap (A\backslash B)|, |S \cap (B\backslash A)|\} = 0$. Then $girth(G^*_\phi) = \infty$.*

Proof. Let $S = \{s_1, \ldots, s_n\}$ where $s_i \notin A \cup B, 1 \leq i \leq n-1$. Without loss of generality, we may assume that $|S \cap (B\backslash A)|\} = 0$. Then $s_n \notin B$. By Britton's Lemma, for any $r \geq 2$, there exists no relation of length less than r among the elements of the generating set

$$S^{(r)} = \{t, t^{-r} s_1 t^{2r}, t^{-3r} s_2 t^{4r}, \ldots, t^{-(2n-3)r} s_{n-1} t^{(2n-2)r}, u^r s_n u^{-2r}\}$$

[6] Britton's Lemma states that in an HNN extension (G, A, B, t), if a word w can be expressed $w = g_0 t^{\epsilon_1} g_1 \ldots \ldots t^{\epsilon_1} g_n, n \geq 1$, with no subwords of the form $t^{-1} g_i t, g_i \in A$, or $t g_j t^{-1}, g_j \in B$, then $w \neq 1$.

where $u = t^{-(2n-3)r}s_{n-1}t^{(2n-2)r}$.

Indeed, in any reduced non-trivial word of length less than r in the alphabet of $S^{(r)}$, written as a reduced word in the alphabet $\{t, s_1, \ldots, s_n\}$, there is no subword of the form $t^{-1}s_n^{\pm 1}t$ hence Britton's lemma still applies. □

The proof of Theorem 1.1 follows from Proposition 2.1, Proposition 2.2, Proposition 3.1 and Proposition 3.2.

Although we are done with the proof of Theorem 1.1, we still would like to prove separately in more explicit terms the fact that a proper HNN extension of D_∞ has infinite girth. Notice that D_∞ is an example of a group when we are unable to separate the generating set S from the union $A \cup B$.

Proposition 3.3. *Let $D_q, q \in (\mathbb{N}\backslash\{0,1\}) \cup \{\infty\}$ be a dihedral group with standard generators a, b, A and B be two proper isomorphic subgroups. Then, girth$((D_q, A, B, t)) = \infty$ for all $q \in (\mathbb{N}\backslash\{0,1\}) \cup \{\infty\}$.*

Proof. We break the proof into the following two cases:

Case 1: If A and B are proper cyclic isomorphic subgroups of D_q, then the hypothesis of Proposition 3.1 can still be arranged. It is easy to see that any proper cyclic subgroup of D_q is isomorphic either to a cyclic group of size q or to $\mathbb{Z}/2\mathbb{Z}$. Let $A = \langle w_1 \rangle; B = \langle w_2 \rangle$ be two proper cyclic subgroups of D_q. Then we have the following sub-cases:

Sub-case 1: For the word length $l(w_1) > 1, l(w_2) > 1$, $S = \{a, b\}$ works, that is $S \cap (A \cup B) = \emptyset$.

Sub-case 2: For $l(w_1) = 1, l(w_2) > 1$, let $w_1 = \{a\}$ then the generating set $S = \{b, ab\}$ does not intersects A and B.

Sub-case 3: For $l(w_1) = 1, l(w_2) = 1$. Choose $S = \{aba, ab\}$.

Hence, we see that the arrangements of Proposition 3.1 still happen in the above sub-cases.

Case 2: A and B are non-cyclic isomorphic subgroups of D_q.

Any non-cyclic subgroup of D_∞ is isomorphic to D_∞; such a subgroup will be of the form $G_{m,n} = \langle a(ba)^m, b(ab)^n \rangle$ where $m, n \in \mathbb{N} \cup \{0\}$. Notice that $G_{0,0} = D_\infty$ and all other subgroups are proper. Similarly, any non-cyclic subgroup of D_q for a finite $q \geq 2$ is isomorphic to D_{q_1} where $q_1|q$, and these subgroups are also of the form $G_{m,n} = \langle a(ba)^m, b(ab)^n \rangle$ for some $m, n \geq 0$. Let $S = \{a, b\}, A = G_{m,n}, B = G_{k,l}$ where $\max\{m, n\} \geq 1$ and $\max\{k, l\} \geq 1$.

If $A \cap S = B \cap S = \emptyset$, then the hypothesis of Proposition 3.1 is satisfied hence $girth((D_q, A, B, t)) = \infty$.

If $A \cap S = \emptyset$ but $B \cap S \neq \emptyset$, then without loss of generality we may assume $b \in B \cap S$. Then $a \notin S$ and replacing S with $S' = \{a, ab\}$ we again satisfy the hypothesis of Proposition 2.1.

The case of $B \cap S = \emptyset$ but $A \cap S \neq \emptyset$ is treated similarly. Thus we are left the case when $A \cap S \neq \emptyset$ and $B \cap S \neq \emptyset$. Then we may assume $a \in A, b \in B$ and $t^{-1}at = u, tbt^{-1} = v$ where $u = a(ba)^m, v = b(ab)^n, m, n \geq 1$. For every $r \geq 1$, we let $S^{(r)} = \{t, t^r at^{-2r}, t^{-r}bt^{2r}\}$. We consider words in the alphabet $t, X = t^r at^{-2r}, Y = t^{-r}bt^{2r}$.

Notice that for all $n \geq 1$,

$$X^n = t^r W_1 t^{-(n+1)r}, X^{-n} = t^{(n+1)r} W_2 t^{-r}, Y^{-n} = t^r W_3 t^{-(n+1)r}, Y^n = t^{(n+1)r} W_4 t^{-r}$$

where $W_i, 1 \leq i \leq 4$ are *suitable* in the sense that it can be written as $W_i = u_1 t^{p_1} u_2 t^{p_2} \dots u_k t^{p_k} u_{k+1}$ with $p_i, 1 \leq i \leq k$ being non-zero integers such that if $u_i \in A$, then $p_{i-1} > 0$ (if $i \geq 1$), $p_i < 0$, and if $u_i \in B$, then $p_{i-1} < 0$ (if $i \geq 1$), $p_i > 0$. Then any word of length less than r in the alphabet $\{t, X, Y\}$ will be still suitable, hence by Britton's Lemma such a word is not identity. □

4. Proof of Proposition 1.2

Given a semi-proper HNN extension $\Gamma = (G, A, B, t)$ with $A = G$, we can form a union $\mathcal{G} = \bigcup_{n \in \mathbb{Z}} t^n G t^{-n}$. Notice that, since $tGt^{-1} = B$, we will have a two-sided infinite chain

$$\dots < t^{-2}Gt^2 < t^{-1}Gt < G < tGt^{-1} < t^2Gt^{-2} < \dots$$

of strict inclusions. Then $\mathcal{G}$ is a normal subgroup and $\Gamma/\mathcal{G} \cong \mathbb{Z}$.

Now, let G be a group satisfying a law. Then G satisfies a law $W(x, y)$ in two variables. To show the claim of Proposition 1.2, we just need to observe that then for all $n \in \mathbb{Z}$, $t^n G t^{-n}$ also satisfies the law $W(x, y)$ hence by the strict inclusions, the normal subgroup $\mathcal{G}$ satisfies $W(x, y)$ as well. On the other hand, the infinite cyclic group $\mathbb{Z}$ also satisfies a law (e.g. $[x, y] = 1$ in $\mathbb{Z}$). It remains to notice that for any short exact sequence $1 \to L \to H \to K \to 1$ of groups, H satisfies a law iff L and K satisfy a law. This finishes the proof of Proposition 1.2.

In the above proof we indeed realized the HNN extension (G, A, B, t) as a semi-direct product $\mathbb{Z} \ltimes_t \mathcal{G}$. The normal subgroup $\mathcal{G}$ is a particularly meaningful object in the case when G is a finitely-generated nilpotent group. For example, if G is also torsion-free, then $\mathcal{G}$ naturally lies inside the Malcev completion of G (see [19]) and one can also treat a general case of a finitely-generated nilpotent group with possibly some torsions.

Let us recall that torsion elements of a nilpotent group G form a subgroup, called *the torsion subgroup*. We will write $Tor(G)$ to denote the torsion subgroup. In addition, if G is finitely generated then the torsion subgroup $Tor(G)$ is finite and normal, moreover, the quotient $G/Tor(G)$ is torsion-free. Now, if we have isomorphic subgroups $A, B \leq G$ with $A = G$, then necessarily $Tor(B) = Tor(A) = Tor(G)$. Hence, for a semi-proper HNN extension (G, A, B, t) where the conjugation by t is given by an isomorphism $\phi : A \to B$ (i.e. by $\phi : G \to B$) then $\phi|_{Tor(G)} : Tor(G) \to Tor(B)$ is an isomorphism and we also obtain an induced isomorphism $\phi_1 : G/Tor(G) \to B/Tor(G)$. Then the group (G, A, B, t) admits a normal subgroup $\mathbb{Z} \ltimes_\phi Tor(G)$ whereas the quotient by this normal subgroup is isomorphic to $(G/Tor(G), A/Tor(G), B/Tor(G), t_1)$ where the conjugation by t_1 is given by the isomorphism ϕ_1.

Let $G_1 = G/Tor(G), A_1 = A/Tor(G), B_1 = B/Tor(G)$. We consider the HNN extension (G_1, A_1, B_1, t_1) given by the isomorphism ϕ_1. Notice that this is a semi-proper HNN extension since $A_1 = G_1$, moreover, G_1 is torsion-free. Now we use the fact that a finitely generated torsion-free nilpotent group H admits a Malcev completion $\overline{H}$ which is also nilpotent of the same nilpotency degree, moreover, any monomorphism $\psi : H \to H$ can be extended to an isomorphism $\overline{\psi} : \overline{H} \to \overline{H}$. Thus the monomorphism $\phi_1 : G_1 \to G_1$ can be extended to an isomorphism $\overline{\phi_1} : \overline{G_1} \to \overline{G_1}$. Then the HNN extension (G_1, A_1, B_1, t_1) is a subgroup of $(\overline{G_1}, \overline{A_1}, \overline{B_1}, t_2)$ where the conjugation by t_2 is given by the isomorphism $\overline{\phi_1}$. Hence the group $(\overline{G_1}, \overline{A_1}, \overline{B_1}, t_2)$ is a semidirect product $\mathbb{Z} \ltimes_{\overline{\phi_1}} \overline{G_1}$. Hence the original HNN extension (G, A, B, t) is a subgroup of a nilpotent extension of a nilpotent group. Since (G, A, B, t) is not infinite cyclic, in particular, we again see that it has a finite girth.

A great example of a Malcev completion can be described for an integral Heisenberg group $H_{\mathbb{Z}} =< x, y|[[x, y], x] = [[x, y], y] = 1 >$. This group is isomorphic to the group $U_3(\mathbb{Z})$ of integral unipotent matrices of size 3×3. The Malcev closure of $H_{\mathbb{Z}}$ will be equal to $U_3(\mathbb{R})$, the group of real unipotent matrices of size 3×3. A semi-proper HNN extension of $H_{\mathbb{Z}}$ will be a subgroup of $\mathbb{Z} \ltimes_\phi U_3(\mathbb{R})$.

However, note that for proper extensions of nilpotent groups this construction fails. Indeed, the following proper HNN extension of $\mathbb{Z}^2 = \langle a, b\rangle$ has infinite girth, which is in support to our Theorem 1.1, $(\mathbb{Z}^2, \langle b^{-1}\rangle, \langle a^n b^{-1}\rangle, t)$ for any $n \in \mathbb{Z}$ with ϕ,

$$\phi : \langle b^{-1}\rangle \to \langle a^n b^{-1}\rangle$$

Then,

$$(\mathbb{Z}^2)^*_\phi = \langle a, b, t|[a, b] = e, t^{-1}b^{-1}t = a^n b^{-1}\rangle = \langle a, b, t|b^{-1}ab = a, b^{-1}tb = ta^n\rangle = \mathbb{F}_2 \rtimes_\phi \mathbb{Z}$$

But, it follows from Proposition 1.4 that

$$girth(\mathbb{F}_2 \rtimes_\phi \mathbb{Z}) = \infty$$

5. Proof of Proposition 1.3

Given some groups A, B, C with monomorphisms $\phi : C \to A$ and $\psi : C \to B$, one can form a product of A and B amalgamated over C. We will write this as $A *_C B$ dropping ϕ and ψ from the notation as they will be given to us in the context. It turns out that A, B, C will have isomorphic images in $G = A *_C B$ which we still denote with the same letters. We will be using the following well known analog of Britton's Lemma for amalgamated free products: Let $T_A = A\backslash C, T_B = B\backslash C$ and $w = g_0g_1 \dots ..g_n, n \geq 1$ such that for all $1 \leq i \leq n$ if

(i) if $g_{i-1} \in A$, then $g_i \in T_B$
(ii) if $g_{i-1} \in B$, then $g_i \in T_A$
(iii) $g_0 \neq 1$
Then $w \neq 1 \in A *_C B$.

We will call the amalgamated free product *proper* if C is a proper subgroup of both A and B. Notice that if $C = A$ ($C = B$) then G becomes isomorphic to B (to A) so $girth(G) = \infty$ iff $girth(B) = \infty$ ($girth(A) = \infty$).

Let $G = A *_C B$ where A, B are finitely generated groups. If C is trivial, then $G = A * B$ and in this simpler case we can proceed as follows: Let $S_1 = \{a_1, \dots, a_n\}, S_2 = \{b_1, \dots, b_m\}$ be generating sets of A and B, respectively, where $1 \notin S_1, 1 \notin S_2$ and any element in the intersection $S_i \cap S_i^{-1}, 1 \leq i \leq 2$ is an involution. If A and B are both cyclic groups then the claim is an easy exercise (alternatively, the group $A * B$ is word hyperbolic hence the result about the girth follows from Theorem 2.6 in [2]), so we will assume that at least one of them, say A, is not cyclic. Then $n \geq 2$. Let now $r \geq 1$. We take reduced words $U_1(X, Y, Z), \dots, U_{m+n}(X, Y, Z), V_1(X, Y, Z), \dots V_{m+n}(X, Y, Z)$ in the free group formally generated by letters X, Y, Z such that for all $1 \leq i, j \leq m+n$ and for some $p > 10r$

(i) U_i ends with Y and V_j begins with Y^{-1};

(ii) $|U_i| = 2p, |V_j| = 2p$, and $|U_i^{\epsilon}V_j^{\delta}| > \frac{7}{2}p$ for all $\epsilon, \delta \in \{-1, 1\}$;

(iii) if $i \neq j$, then $|U_i^{\epsilon}U_j^{\delta}| > \frac{7}{2}p$ and $|V_i^{\epsilon}V_j^{\delta}| > \frac{7}{2}p$ for all $\epsilon, \delta \in \{-1, 1\}$;

(iv) U_i and V_j are reduced words of length p in the alphabet $\{\xi, \eta\}$ where $\xi = XY, \eta = ZY$;

(v) if $W(\xi, \eta)$ is any reduced word of length at most r, then for all $\epsilon, \delta \in \{-1, 1\}$, the inequality $\min\{|U_i^{\epsilon}WU_j^{\delta}|', |U_i^{\epsilon}WV_j^{\delta}|', |V_i^{\epsilon}WV_j^{\delta}|'\} > \frac{3}{2}p$ holds

where $|.|$ and $|.|'$ denote the Cayley metrics with respect to the alphabets $\{X, Y, Z\}$ and $\{\xi, \eta\}$ respectively.

Let us observe that any reduced word in the alphabet $\{\xi, \eta\}$, when viewed as a reduced word in the alphabet $\{X, Y, Z\}$ does not contain any of the subwords X^2, Y^2, Z^2.

Now we let $S^{(r)} = S_1^{(r)} \cup S_2^{(r)} \cup \{uv, wv\}$ where

$$S_1^{(r)} = \{U_1(u, v, w)a_1V_1(u, v, w), \dots, U_n(u, v, w)a_nV_n(u, v, w)\}$$

and

$$S_2^{(r)} = \{U_{n+1}(u, v, w)a_1b_1a_1V_{n+1}(u, v, w), \dots, U_m(u, v, w)a_1b_ma_1V_m(u, v, w)\}$$

where $u = a_1b_1a_1^{-1}, v = a_2b_1a_2^{-1}, w = a_1a_2b_1a_2^{-1}a_1^{-1}$.

By condition (iv) we have that $U_i, V_i, 1 \leq i, j \leq m+n$ are also words of length p in uv and wv. Then $S^{(r)}$ is a generator of $G = A * B$. Notice that none of the words

$$a_1, a_2, a_1^{-1}a_2, a_2a_1^{-1}, a_1^{-1}a_2a_1, a_1a_2a_1^{-1}, a_2^{-1}a_1a_2, a_2a_1a_2^{-1}$$

represents identity element in A[7], hence there is no relation of length less than r among the elements of $S^{(r)}$. Thus, $girth(A * B, S^{(r)}) \geq r$.

In the general case, when $C \neq 1$, by the index assumption of our proposition, without loss of generality, we may assume that $A : C \geq 3$.

We will use the following simple lemma.

Lemma 5.1. *Let G be a group and $H \leq G$ be a subgroup such that for all $x \in G \backslash H, x^2 \in H$. Then H is a normal subgroup.*

Proof. Indeed, let $h \in H, x \in G \backslash H$. $xh \in G \backslash H$ hence $(xh)^2 \in H$ which yields $xhx \in H$. Then $xhx^{-1} = (xhx)x^{-2} \in H$. □

Using Lemma 5.1, we can claim another simple lemma.

Lemma 5.2. *Let A be a group and $C \leq A$ with $A : C \geq 3$. Then there exists distinct $a_1, a_2 \in A$ such that $a_1, a_2, a_1^{-1}a_2, a_2a_1^{-1} \notin C$ and $a_1^{-1}a_2a_1, a_1a_2a_1^{-1}, a_2^{-1}a_1a_2, a_2a_1a_2^{-1} \notin C$.*

Proof. If C is not trivial and there exists $x \in A \backslash C$ such that $x^2 \notin C$, then we can take $a_1 = x, a_2 = x^2$; otherwise, by Lemma 5.1, C is a normal subgroup thus we just need to find two distinct elements $d_1, d_2 \in (G/C) \backslash \{1\}$ satisfying conditions $d_1^{-1}d_2 \neq 1$ but such elements trivially exist if $|G/C| \geq 3$. □

[7]This also implies that if b_1 is not a torsion element, then u, v, w generate a free subgroup of rank 3 in $G = A * B$; so, in the case when b_1 is not a torsion, we could present an easier argument.

Then we can choose generating sets $S_1 = \{a_1, \ldots, a_n\}, S_2 = \{b_1, \ldots, b_m\}$ of A and B such that $S_1 \cap C = S_2 \cap C = \emptyset$, $n \geq 2$ and none of the elements

$$a_1, a_2, a_1^{-1}a_2, a_2a_1^{-1}, a_1^{-1}a_2a_1, a_1a_2a_1^{-1}, a_2^{-1}a_1a_2, a_2a_1a_2^{-1}$$

belong to C. Then we define $S^{(r)}$ as above and there will be no relation of length less than r among the elements of $S^{(r)}$. Thus we again obtain that $girth(A *_C B, S^{(r)}) \geq r$. Since r is arbitrary, we conclude that $girth(A *_C B) = \infty$.

Remark 5.3. Notice that a proper amalgamated free product $G = A *_C B$ is virtually solvable if and only if C is a virtually solvable normal subgroup and $G/C \cong D_\infty$. However, when C is a non-virtually solvable normal subgroup satisfying a law and $G/C \cong D_\infty$, the girth of G is still finite. So, for the class of proper amalgamated free products, Girth Alternative cannot be extended beyond Proposition 1.3 to cover the case $A : C = B : C = 2$.

6. Proof of Proposition 1.4

Let $\Gamma = (G, A, B, t)$ be an HNN extension of a non-elementary word hyperbolic group G.

For proper HNN extension (G, A, B, t), with $\phi : A \to B$ an isomorphism between proper subgroups $A, B < G$, the claim follows from Theorem 1.1 that $girth((G, A, B, t)) = \infty$.

We will treat the cases of proper and full HNN extensions together. As a major tool, we will consider the actions of word hyperbolic groups on their boundary. Let us recall that the boundary of a word hyperbolic group G is a compact metric space, denoted as ∂G. G acts on ∂G by homeomorphisms. Torsion elements of $G \backslash \{1\}$ are called *elliptic* and non-torsion elements are called *hyperbolic*. A hyperbolic element g has exactly two fixed points on ∂G, one of them is attractive and another one is repelling; we will denote these as P_g and R_g respectively.

We will use the following proposition (See [18]).

Proposition 6.1. *Let G be a non-elementary word hyperbolic group. Then*
a) The boundary ∂G is infinite;
b) The sets $\{P_g : g \text{ is a hyperbolic element of } G\}$ *and* $\{R_g : g \text{ is a hyperbolic element of } G\}$ *are dense in ∂G;*
c) The set $\{(P_g, R_g) : g \text{ is a hyperbolic element of } G\}$ *is dense in* $\partial G \times \partial G$.

For convenience of the reader we will run the argument for word hyperbolic groups and then extend it to HNN extensions. The following proposition reproves a result (Theorem 2.6) of [2]. Making use of Proposition 6.1, we offer a simpler proof of this result.

Proposition 6.2. *A non-elementary word hyperbolic group has infinite girth.*

Proof. Let $\{g_1, \ldots, g_s\}$ be a finite generating set of a word hyperbolic group G and γ be a hyperbolic element of G. By Proposition 6.1, there exists a hyperbolic element $\beta \in G$ such that

$$(Fix(\beta) \cup \bigcup_{1 \leq i \leq s} g_i^{\pm 1} Fix(\beta)) \cap (Fix(\gamma) \cup \bigcup_{1 \leq i \leq s} g_i^{\pm 1} Fix(\gamma)) = \emptyset.$$

Since ∂G is a compact metric space, it is Hausdorff, so we can take open neighborhoods U, V of $Fix(\gamma), Fix(\beta)$ respectively such that

$$(U \cup \bigcup_{1 \leq i \leq s} g_i^{\pm 1} U) \cap (V \cup \bigcup_{1 \leq i \leq s} g_i^{\pm 1} V) = \emptyset$$

and the set

$$\partial G\backslash\overline{((U \cup \bigcup_{1\leq i\leq s} g_i^{\pm 1}U) \cup (V \cup \bigcup_{1\leq i\leq s} g_i^{\pm 1}V))}$$

is infinite. Let P be any point in this set.

Since the fixed points of γ and β are either attractive or repelling, there exists a natural N such that for all $n \geq N, \gamma^{\pm n}(\partial G\backslash U) \subseteq U$ and $\beta^{\pm n}(\partial G\backslash V) \subseteq V$.

Then for every $r \geq 1$ we can take a generating set

$$S_r = \{\beta^N, \gamma^N, \beta^{Nr}g_1\gamma^{Nr}, \beta^{2Nr}g_2\gamma^{2Nr}, \dots, \beta^{sNr}g_s\gamma^{sNr}\}.$$

By our arrangements, for any word W in this alphabet, we will have $W(P) \in U \cup V$, hence $W(P) \neq P$, hence $W \neq 1$. This implies that $girth(G) \geq r$. Since r is arbitrary, we conclude that $girth(G) = \infty$. □

Now we are ready to discuss the HNN extensions of word hyperbolic groups.

Let again $r \geq 1$ and $\{g_1, \dots, g_s\}$ be a finite generating set of a word hyperbolic group G. Again, we can choose hyperbolic elements β, γ such that the sets

$$A(\beta) = (\bigcup_{0\leq j\leq 2r} Fix(t^{-j}\beta t^j)) \cup (\bigcup_{1\leq i\leq s, 0\leq j,l\leq 2r} t^{-j}g_i^{\pm 1}t^j Fix(t^{-l}\beta t^l))$$

and

$$A(\gamma) = (\bigcup_{0\leq j\leq 2r} Fix(t^{-j}\gamma t^j)) \cup (\bigcup_{1\leq i\leq s, 0\leq j,l\leq 2r} t^{-j}g_i^{\pm 1}t^j Fix(t^{-l}\gamma t^l))$$

are disjoint.

By letting $g_0 = 1$ we can conveniently write

$$A(\beta) = \bigcup_{0\leq i\leq s, 0\leq j,l\leq 2r} t^{-j}g_i^{\pm 1}t^j Fix(t^{-l}\beta t^l) \text{ and } A(\gamma) = \bigcup_{0\leq i\leq s, 0\leq j,l\leq 2r} t^{-j}g_i^{\pm 1}t^j Fix(t^{-l}\gamma t^l).$$

The above arrangement will allow us to write $\beta^i\gamma^j$ and $\gamma^j\beta^i$ as $\beta^i g_0\gamma^j$ and $\gamma^j g_0\beta^i$ respectively.

We can take open neighborhoods U, V of $\bigcup_{0\leq j\leq 2r} Fix(t^j\beta t^{-i})$ and $\bigcup_{0\leq j\leq 2r} Fix(t^j\gamma t^{-i})$ respectively such that

$$(U \cup \bigcup_{1\leq i\leq s, 0\leq j\leq 2r} (t^{-j}g_i^{\pm 1}t^j)U) \cap (V \cup \bigcup_{1\leq i\leq s, 0\leq j\leq 2r} (t^{-j}g_i^{\pm 1}t^j)V) = \emptyset$$

and the complement

$$\partial G\backslash\overline{(U \cup \bigcup_{1\leq i\leq s, 0\leq j\leq 2r} (t^{-j}g_i^{\pm 1}t^j)U) \cup (V \cup \bigcup_{1\leq i\leq s, 0\leq j\leq 2r} (t^{-j}g_i^{\pm 1}t^j)V)}$$

is infinite. Again, taking an arbitrary point P in this complement, for sufficiently big N, we observe that there is no relation of length less than r among the elements of

$$S = \{t, \beta^N\gamma^N, \gamma^N(\beta^N\gamma^N)^r, \beta^{Nr}g_1\gamma^{Nr}, \dots, \beta^{sNr}g_s\gamma^{sNr}\}.$$

Indeed, if W is a such a relation, since Γ is a semi-proper or full extension [8], by Britton's Lemma, the sum of exponents of t in W is equal to zero and a cyclic permutation W' of W or W^{-1} can be written as

$$W' = t^q(t^{-i_1}\beta^{N_1}t^{i_1})(t^{-j_1}g_{l_1}t^{j_1})(t^{-k_1}\gamma^{M_1}t^{k_1})\dots(t^{-i_m}\beta^{N_m}t^{i_m})(t^{-j_m}g_{l_m}t^{j_m})(t^{-k_m}\gamma^{M_m}t^{k_m})t^{-q}$$

[8]In a semi-proper or full HNN extension (G, G, B, t), every element can be written as $t^p g t^{-q}$ for some $g \in G$ and non-negative integers p, q.

where $l_1, \dots, l_m \in \{0, 1, \dots, s\}$, all exponents of t (i.e. the numbers $i_1, j_1, k_1, \dots, i_m, j_m, k_m$) are non-negative, moreover, the exponents $i_1, j_1, k_1, \dots, i_m, j_m, k_m$ are less than r, and the exponents $N_1, M_1, \dots, N_m, M_m$ are bigger than N in absolute value. (It is also important to recall that we let $g_0 = 1$). Then, by our arrangement, $W'(p) \neq p$, hence $W' \neq 1$, hence $W \neq 1$.

Thus $girth(\Gamma) \geq r$ and since r is arbitrary, we obtain that $girth(\Gamma) = \infty$. This completes the proof of Theorem 1.4.

In the case of free groups of rank at least two (i.e. $G \cong \mathbb{F}_n, n \geq 2$), it is not straightforward to find direct elementary proofs. Much studies have been done in recent years about full or semi-proper HNN extensions of non-abelian free groups, yet Corollary 1.5 does not seem to lend itself easily to these results either.

Remark 6.3. Let us emphasize that free-by-cyclic groups $\mathbb{Z} \ltimes_\phi \mathbb{F}_n$ are not always hyperbolic so we cannot invoke the result from [2] (Theorem 2.6) about hyperbolic groups. In fact, by the result of P.Brinkmann [7] these groups are hyperbolic precisely when they are atoroidal, i.e. when they do not contain an isomorphic copy of $\mathbb{Z} \oplus \mathbb{Z}$. Moreover, for $n = 2$, $\mathbb{Z} \ltimes_\phi \mathbb{F}_n$ is always toroidal, hence not hyperbolic. Hyperbolicty does not hold for semi-proper HNN extensions either, in fact, a result of [20] states that a semi-proper HNN extension of $\mathbb{F}_n$ is hyperbolic unless it contains a copy of Baumslag-Solitar group $BS(1, m)$ for $m \geq 1$.

Remark 6.4. One can hope to invoke Theorem 4.4 from [2] but linearity is also a problematic issue. It is not known whether free-by-cyclic groups are linear, although this is known to be true in the case when the rank of the free group is two [11]. For semi-proper HNN extensions, the presentation $\langle t, a, b \mid t^{-1}at = a^m, t^{-1}bt = b^r \rangle$ defines a well-known example studied in [23] and [16] which, for $m, r \geq 2$, gives a non-linear group.

It is known that free-by-cyclic groups are either isomorphic to $BS(1, \pm 1)$, or hyperbolic, or large [12] (see also [13] where it is shown that a non-solvable free-by-cyclic group is large) and for semi-proper HNN extensions of a free group, it has been conjectured by J.Button (in personal communications) that any such group is either hyperbolic or large, unless it is isomorphic to $BS(1, m), m \in \mathbb{Z} \backslash \{0\}$ (a group is large if it has a finite index subgroup which surjects onto a non-abelian free group). This suggests a use of another result from [2], namely, Proposition 1.1 (combined with Theorem 2.6 there), but again, we have an issue of not being able to deduce infinity of the girth of a group from the infinity of the girth of its finite index subgroup. Indeed, we would like to ask the following

Question 1: Do finitely generated large groups have infinite girth?

The following general question regarding full HNN extensions of a given group G is also interesting to us:

Question 2: Let G be a non-cyclic finitely generated group. Is it true that $girth(G) = \infty$ if and only if $girth(\mathbb{Z} \ltimes_\phi G) = \infty$.

If $girth(G) = \infty$, then, in the case of trivial $\mathbb{Z}$-action (i.e. the direct sum), since G is a quotient of $\mathbb{Z} \ltimes_\phi G$, by Proposition 1.1.in [2], we obtain that $girth(\mathbb{Z} \ltimes_\phi G) = \infty$. The case of general $\mathbb{Z}$-action remains unclear. We conjecture that in the other direction the answer is negative, i.e. there exists a finitely generated group G and an automorphism ϕ such that

$girth(G) < \infty$ whereas $girth(\mathbb{Z} \ltimes_\phi G) = \infty$. Let us also mention (a somewhat relevant fact) that if G is a finitely generated group that does not satisfy a law in two variables with $girth(G) < \infty$, then $girth(((G \wr \mathbb{Z}) \wr \mathbb{Z})) = \infty$ as shown in [1].

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