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Orientability of smooth manifolds

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Abstract

This semi-expository note provides a pedagogical account of the notion of orientability in smooth manifolds. We describe various settings in which orientability is established and used. Sections 5, 11, and 12 contain original results. In these and other sections, we bring together various results that are otherwise buried in the literature. Section 5 discusses the orientability of manifolds with a finite atlas in terms of a finite graph associated with the given atlas. In Section 11, we determine which orientable 3-manifolds cover a non-orientable one, depending on which of the eight geometries (in the sense of Thurston) they admit. Section 12 is devoted to the question of when a given finitely presented group occurs as a fundamental group of a closed *non-orientable* manifold, and we provide both positive and negative results.

Keywords: smooth manifolds, orientable manifolds, oriented atlas.

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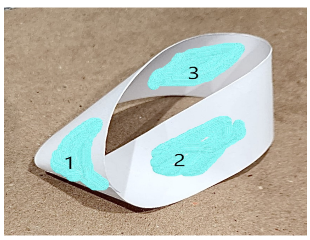
Contents

1 Introduction

Orientability is introduced to students very early on, even before a formal introduction to topology. For circle and real line, this is done by choosing a direction on them (sometimes even without using the term "orientation"). For surfaces, we can think of orientability as two-sidedness. A sphere has two sides, whereas a Möbius band has only one. Starting somewhere on the sphere, by coloring it, we will end up with only one side of it colored; we can then start coloring the other side using a different color. Thus, the surface of our sphere will be colored using two colors and the colors never meet each other, one remains "inside" while the other color will be on the "outside".

This is not the case with the Möbius band; if we start coloring anywhere on the surface of it (with a blue paint, on the white background), we will end up coloring "all of it". In dimension three and higher (i.e. for n-dimensional manifolds with $n \geq 3$) the notion of orientation becomes more obscure and we feel we are in need of a rigorous definition.

Although contentful and interesting, the two-dimensional definition has numerous problems. It lacks rigour, first of all, moreover, although the definition is very intuitive, we encounter significant difficulties in trying to adapt it to a rigourous language. Understanding orientability as two-sidedness seems to be the right idea, but what is a side? If we test this definition in the 1-dimensional case, we encounter a confusing picture: would not the 1-dimensional manifolds



(circle and line) have only one side? (i.e. can't we color them in just one color?) The terms "inside" and "outside" in the previous paragraph were useful but it seems that then we are using embedded manifolds here (i.e. embedding a surface (2-manifold) in $\mathbb{R}^3$), though ideally we would prefer an intrinsic definition. Nevertheless, the idea of using an embedding seems attractive. For example, an embedded circle or an embedded line in a plane [1] bounds a region whereas we know that non-orientable 2-manifolds such as the real projective plane or the Klein bottle do not bound a region in $\mathbb{R}^3$, in fact, do not even embed in $\mathbb{R}^3$; these manifolds embed in $\mathbb{R}^4$ but do not bound a region there either. On the other hand, all the embedded *orientable* 2-manifolds in $\mathbb{R}^3$ bound a region. So, does bounding a region have anything to do with orientability?

One purpose of this paper is to make orientability a less mysterious notion to aspiring topologists, and to reveal some of its usefulness. The first part of the paper covers some of the basic definitions and properties of orientability, answering the questions raised above (see §2.3) and giving a constructive approach to test orientability (Theorem 4.2). In the second part, we investigate some of the geometric and algebraic implications of orientability to three manifolds and non-orientable manifolds.

Orientability is a topological invariant. Its non-existence prevents the existence of many geometric structures on manifolds. For example, non-orientable manifolds can never be complex or symplectic manifolds (see Sections 7 and 8). They can never be Lie groups, and if closed, they can never be embedded in a Euclidean space of one dimension higher. Moreover, orientability is intimately related to the intersection theory on the manifold; in an oriented manifold M, if two oriented submanifolds A and B of complementary dimensions intersect transversely, they define an integer intersection number $A \cdot B \in \mathbb{Z}$, giving a geometric understanding of the cup product dually. Using orientability, one can also define a fundamental notion of a degree of mappings between closed orientable manifolds of the same dimension, assigning an integer to a mapping! The use of orientation on manifolds goes even beyond mathematics; it has profound implications in modern physics.

Because of the many reasons listed above, and of our own experience teaching this material at North Dakota State University, we thought that a semi-expository note on this material in GJM would be helpful to beginner, or even more advanced graduate students. As the title suggests, these notes focus on orientability of *smooth* manifolds. A topological manifold of dimension $n > 3$ does not necessarily admit a smooth structure, and if it does, it may not be unique. In dimensions one, two, and three, smooth structures always exist and are unique. The definition of an orientation in the non-smooth setting requires homology theory and we refer to Chapter VIII of [6] for a thorough treatment. Our methods are overwhelmingly geometric.

[1] For a topological space X, we call a continuous map $F : X \to \mathbb{R}^n$ an embedding, if $f(X)$ is a closed subset of $\mathbb{R}^n$ and f is a homeomorhism in between X and $f(X)$

2 Orientability for smooth manifolds

2.1 Orientation of $\mathbb{R}^n$

A manifold[2] is locally $\mathbb{R}^n$ for some $n \geq 1$, hence we will first define an orientation for the space $\mathbb{R}^n$ and then work our way out to extend this notion to general manifolds.

Let n be a positive integer and let $(u_1, \dots, u_n)$ and $(v_1, \dots, v_n)$ be any two bases of $\mathbb{R}^n$. We want to find an isotopy $\Phi : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ that transforms the first basis to the second one, and at any time $t \in [0,1]$, the n-tuple $(\Phi(u_1, t), \dots, \Phi(u_n, t))$ remains a basis.

For $n = 1$, clearly this is not always possible. In fact, two non-zero vectors u and v will be isotopic this way if and only if they have the same sign (in dimension one, we can identify vectors with numbers, so vectors will have signs). For a general n, again we have only two isotopy classes of bases: $(u_1, \dots, u_n)$ and $(v_1, \dots, v_n)$ will be isotopic if and only if there exists a matrix $A \in GL(n, \mathbb{R})$ such that $\det(A) > 0$ and $Au_i = v_i, 1 \leq i \leq n$. For example, if we take the standard basis $\mathcal{B} = (e_1, \dots, e_n)$ of $\mathbb{R}^n$, and consider any permutation $\pi(\mathcal{B}) = (\pi(e_1), \dots, \pi(e_n))$, then $\mathcal{B}$ and $\pi(\mathcal{B})$ will be isotopic if and only if the permutation π is even. For $n \geq 2$, any basis will be isotopic either to $(e_1, e_2, \dots, e_n)$ or to $(e_2, e_1, \dots, e_n)$. It is also true that for $n \geq 1$, any basis will be isotopic either to $(e_1, e_2, \dots, e_n)$ or to $(-e_1, e_2, \dots, e_n)$. Orienting $\mathbb{R}^n$ is, indeed, about fixing the isotopy class of a basis. Since we have two equivalence classes up to the isotopy, we have two ways to choose our orientation.

Notice that the subgroup

$$GL_+(n, \mathbb{R}) = \{C \in GL(n, \mathbb{R}) : \det(C) > 0\}$$

is the connected component of identity in the Lie group $G = GL(n, \mathbb{R})$. The discussion above

[2] "Manifold" will refer to a manifold *without* boundary unless we explicitly say otherwise. The term *closed manifold* will always refer to a compact manifold without a boundary.

is precisely the statement that $GL(n,\mathbb{R})$ has exactly two connected components: $GL_+(n,\mathbb{R})$ and its complement. A basis of $\mathbb{R}^n$ is positively oriented if the matrix of basis vectors it defines belongs to $GL_+(n,\mathbb{R})$.

2.2 Charts and Orientation

Defining the notion of orientation, and thus orientability, is easier if a manifold admits a smooth structure. The reason is that for smooth maps $F : D \to \mathbb{R}^n$, where D is a domain in $\mathbb{R}^n$, one has the notion of the Jacobian J_xF, which is an $n \times n$-matrix for every $x \in D$. For such a map F, we say that it preserves the orientation if $\det J_pF > 0$ for all $p \in D$. Then, if $(e_1, \dots, e_n)$ is the standard basis, at any point $p \in D$, the differential map $D_pF : \mathbb{R}^n \to \mathbb{R}^n$ takes it to another basis $(D_pe_1, \dots, D_pe_n)$ such that the matrix $A := J_pF$ mapping one n-tuple to another has a positive determinant. Notice that in this case, if we choose any permutation $(\pi(e_1), \dots, \pi(e_n))$, then the matrix taking it to $(D_p\pi(e_1), \dots, D_p\pi(e_n))$ is conjugate to A, and hence its determinant will also be positive. So, if F preserves an orientation, then it preserves *the other orientation* as well.

Definition 2.1. Let $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}_{\alpha\in I}$ be a smooth atlas [3] with the transition maps $\psi_{\alpha\beta} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta), \alpha, \beta \in I$. We say that $\mathcal{U}$ is an *oriented atlas* if for all $\alpha, \beta \in I$ with $U_\alpha \cap U_\beta \neq \emptyset$, $\psi_{\alpha\beta}$ is orientation-preserving. M is called *orientable* if it admits an oriented atlas. M is called *oriented* if we have already fixed an oriented atlas on M.

If we try to use the above definition to investigate $\mathbb{S}^n$ or $\mathbb{RP}^n$, we will see that even if we choose the "easiest possible" atlas, computing transition maps and their Jacobians requires a good amount of work. Since the orientability, or lack thereof, of projective spaces is one of the most illustrative early examples, it is treated in some detail below, and is a strongly recommended exercise.

Example 2.2. *The real projective space $\mathbb{RP}^n$ is orientable for all odd n.*

Let us recall that for $n = 1$, $\mathbb{RP}^1$ is diffeomorphic to $\mathbb{S}^1$ and for $n = 3$, $\mathbb{RP}^3$ is diffeomorphic to $SO(3)$, so in both cases our space is diffeomorphic to a Lie group. We will see later that Lie groups are always orientable, but for odd $n \geq 5$ (and for all even n), $\mathbb{RP}^n$ does not admit a Lie group structure. This is because a cover of a Lie group is still a Lie group, but the manifold $\mathbb{S}^n$, as 2-sheeted cover of the $\mathbb{RP}^n$, admits a Lie group structure only for $n = 0, 1$ and 3.

For an arbitrary n, $\mathbb{RP}^n$ admits a finite atlas consisting of $n+1$ charts $(U_i, \phi_i), 0 \leq i \leq n$ with open sets

$$U_i = \{[X_0 : X_1 : \cdots : X_n] : X_i \neq 0\}, 0 \leq i \leq n$$

and $\phi_i(X_0 : X_1 : \cdots : X_n) = (\frac{X_0}{X_i}, \dots, \widehat{\frac{X_i}{X_i}}, \dots, \frac{X_n}{X_i})$. Unfortunately (and interestingly) this particular atlas is not an oriented one. Indeed, for $1 \leq i < j \leq n$, computing the transition maps $\psi_{ij} = \phi_j \circ \phi_i^{-1}$, we obtain that

$$\begin{aligned} &\psi_{ij}(t_1, \dots, t_n) \\ = \ &(\frac{t_1}{t_j}, \dots, \frac{t_i}{t_j}, \frac{1}{t_j}. \frac{t_{i+1}}{t_j}, \dots, \frac{t_{j-1}}{t_j}, \frac{t_{j+1}}{t_j}, \dots, \frac{t_n}{t_j}) \end{aligned}$$

where $(t_1, \dots, t_n) = (\frac{X_0}{X_i}, \dots, \widehat{\frac{X_i}{X_i}}, \dots, \frac{X_n}{X_i})$.

Computing the determinant of the Jacobian, we get

$$\det(J\psi_{ij}(t_1, \dots, t_n)) = \frac{(-1)^{j-i}}{t_j^{n+1}}.$$

As we cautioned above, this determinant is not always positive. However, if we replace our original atlas $\mathcal{U} = \{(U_i, \phi_i) : 0 \leq i \leq n\}$ with $\mathcal{U}' = \{(U_i, (-1)^i\phi_i) : 0 \leq i \leq n\}$, then we will end up with the expression $\frac{(-1)^{(n+1)j}}{t_j^{n+1}}$ as the determinant of the Jacobian of the (ij)-th transition map. This expression is always positive for odd n. Hence, for odd n, $\mathbb{RP}^n$ admits an oriented atlas $\mathcal{U}'$ therefore, it is orientable. To show non-orientability of $\mathbb{RP}^n$ for even n, we need to come up with a new argument. In the most standard way, this can be done by invoking homology groups. For a closed manifold, the top homology group is either isomorphic to $\mathbb{Z}$ or trivial depending on whether the manifold is orientable or not. The homology group $H_n(\mathbb{RP}^n)$ is trivial for even n (and it is infinite cyclic for odd n).

2.3 Orientability for Hypersurfaces

Exploring Definition 2.1 allows us to simplify the notion of orientability for smoothly embedded surfaces in $\mathbb{R}^3$, and, more generally, for smoothly embedded n-manifolds in $\mathbb{R}^{n+1}$. In fact, having an n-manifold embedded in the space with

[3] We will typically work with atlases $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in I}$ where the range $\phi_\alpha(U_\alpha)$ is an open ball in $\mathbb{R}^n$ or the entire $\mathbb{R}^n$, although our definition allows $\phi_\alpha(U_\alpha)$ to be an open subset of $\mathbb{R}^n$. Sometimes we will denote atlases as $\{(U_\alpha, V_\alpha, \phi_\alpha)\}_{\alpha\in I}$ where $V_\alpha = \phi_\alpha(U_\alpha)$. For basic elements of the theory of differentiable manifolds, we refer the reader to [14].

one dimension higher provides a suitable environment in understanding orientability and in trying to establish it, if possible.

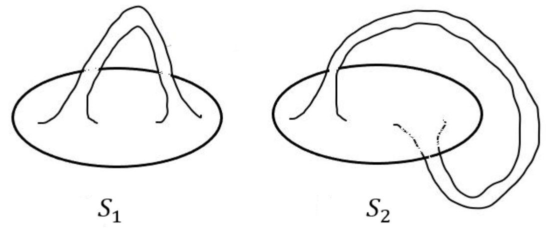
Let M be a smooth submanifold of $\mathbb{R}^{n+1}$ with $\dim M = n$. At every point $p \in M$, we have the tangent space T_pM as a hyperplane in $\mathbb{R}^{n+1}$. Then there exist exactly two unit normal vectors at p. An orientation on M is the choice of a unit normal vector field on M that is *continuous*. Notice that if we have such a vector field, then by replacing a vector with its additive inverse at each point, we get another continuous unit normal vector field. Then, it is straightforward to see that these are the only two continuous unit normal vector fields on M.

For a smoothly embedded surface $S \subset \mathbb{R}^3$ let us fix a continuous unit normal vector field on it (assuming that such a field exists). Then this vector field can be viewed as *a side of* S. The other side is obtained by choosing the opposite vector field. Thus, orientability gives rise to the notion of side. Conversely, if S bounds a domain in $\mathbb{R}^3$, then this allows defining a side for S (the inside and the outside) and this can be used (see Section 6) to put an orientation on S.

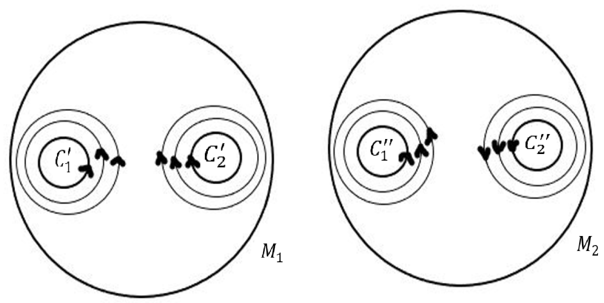
In general, we have the following fundamental result: *any hypersurface of* $\mathbb{R}^{n+1}$ *must be orientable*. A very nice argument of proof is given by Samelson [18] and goes as follows: if M is a smooth n-dimensional *non-orientable* submanifold of $\mathbb{R}^{n+1}$, then by taking a point on the normal a small distance off M and moving it around the loop and then connecting along the normal from one side of M to the other, we can get a simple closed C^∞-curve γ meeting our manifold M at exactly one point p such that γ is transversal to M at p. On the other hand, the loop γ is contractible in $\mathbb{R}^{n+1}$. Then, we have a smooth map $f : \mathbb{D}^2 \to \mathbb{R}^{n+1}$ such that at the boundary $\partial\mathbb{D}^2$, f yields γ, moreover, f is transversal to M (so $f(\mathbb{D}^2)$ is not tangent to M at any common point). By transversality, the preimage $f^{-1}(M)$ consists of closed C^∞-curves in $\mathrm{Int}\mathbb{D}^2$ and C^∞-arcs that meet $\partial\mathbb{D}^2$ at the endpoints (which are distinct!). But this is impossible because the intersection of γ with M consists of just one point.

3 Cutting a disk from a 2-manifold

The torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ and the Klein bottle $\mathbb{K}$ have a special partnership. Although easily distinguishable by the basic tools of algebraic topology (such as π_1, H_1, H_2), the two manifolds also share many properties: both manifolds have Euler characteristic zero being the only two closed 2-manifolds with zero Euler characteristic. Torus is a double cover of Klein bottle, and both manifolds are covered by the Euclidean plane; indeed, torus and Klein bottle are the only two Euclidean closed 2-manifolds. Both manifolds are affine, and again, these are the only two closed affine manifolds in dimension two.



Let us cut off an open disk from both manifolds. Any 2-dimensional closed manifold, after cutting off a disk of it, becomes embeddable in $\mathbb{R}^3$. For torus and Klein bottle, we obtain surfaces $S_1 = \mathbb{T}^2 \backslash D^2$ and $S_2 = \mathbb{K} \backslash D^2$ as in the picture. Both surfaces are made up from the same "materials": a disk D with smaller disjoint disks D_1, D_2 cut off, a cylinder $\mathbb{S}^1 \times I$ with boundary circles glued to the boundary circles ∂D_1 and ∂D_2. It might seem at first that the surfaces S_1 and S_2 must be homeomorphic. In fact, sending the cylinders (the handles) of S_1 to S_2 identically leaves us finding a homeomorphism of the surfaces M_1 and M_2 where we need to glue the *parametrized circles* C_1' to C_2' and C_1'' to C_2'' as indicated in the figure below.



However, it turns out that this is impossible because of the way the circle C_2'' is oriented. If C_2' is glued to C_2'', then this will force us to glue the circles around C_2' to the circles around C_2'' so that the orientation matches. On the other hand, we are also forced to glue the circles around C_1' to the circles around C_2' again, matching the orientation. When the circles expanded around C_2', as we try to build our map, they start meeting the circles expanded around C_2'', we encounter a mismatch in how these two sets of concentric circles are oriented; this does not allow us to build a homeomorphism $M_1 \to M_2$ as indicated. Such a homeomorphism is in fact impossible; its existence would imply that $S_1 \cong S_2$, however, S_1 is an orientable surface (it is two-sided) whereas S_2 is not orientable (it is one-sided).

In general, if M is an n-dimensional manifold and B is a disjoint ball in M, then $M \backslash B$ is orientable if and only if M is. Cutting off a disk from the torus and Klein bottle, we obtain orientable and non-orientable surfaces respectively. Thus, albeit at a somewhat intuitive level, we are able to see the non-homemorphism of the torus and Klein bottle using just the notion of orientability.

4 Atlases with finitely many charts

Immediately from the definition of orientability, we deduce that if a manifold admits an atlas with just one open set, then it is orientable. This is not surprising because then our manifold will be connected and can be identified with a domain in $\mathbb{R}^n$; and the latter can take its orientation from $\mathbb{R}^n$.

Proposition 4.1. *Let M be a smooth manifold admitting and atlas $\mathcal{U}$ consisting of just two charts (U_1, ϕ_1) and (U_2, ϕ_2). Let also the intersection $U_1 \cap U_2$ be connected. Then the manifold M is orientable.*

For example, a sphere $\mathbb{S}^n$ admits such an atlas; it is, in fact, the most common atlas of it: assuming the "equator" of $\mathbb{S}^n$ is the zeroth "parallel", for any sufficiently small $\epsilon > 0$, taking the region above the parallel $-\epsilon$ and the region below the "parallel" ϵ, we obtain such an atlas. For the projective space $\mathbb{RP}^n$, it is not obvious whether such an atlas exists, even for $n = 2$. We know that the standard atlas of $\mathbb{RP}^n$ consists of $n+1$ open sets

$$U_i = \{[X_0 : X_1 : \cdots : X_n] : X_i \neq 0\}, 0 \leq i \leq n.$$

Any n of these open sets have a non-empty intersection, but no point of the space belongs to all of them.

We now want to prove our claim in Proposition 4.1. Let us fix a map $\psi : \mathbb{R}^n \to \mathbb{R}^n$ defined as $\psi(x_1, x_2, \ldots, x_n) = (-x_1, x_2, \ldots, x_n)$. ψ is a reflection, it reverses an orientation of $\mathbb{R}^n$ (ψ is a linear map and its matrix representation with respect to any choice of a basis of $\mathbb{R}^n$, with the same choice in the domain and co-domain, will have a negative determinant). Since we have only two charts, we have only two transition maps

$$\psi_{1,2} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2) \text{ and}$$
$$\psi_{2,1} : \phi_2(U_1 \cap U_2) \to \phi_1(U_1 \cap U_2).$$

Notice that for all $z \in U_1 \cap U_2$, we have

$$\det(J\psi_{1,2}(\phi_1(z))) = \det(J\psi_{2,1}(\phi_2(z)))^{-1}$$

thus, it suffices to verify the orientability condition only for the transition map $\psi_{1,2}$.

Let $x \in U_1 \cap U_2$ such that for the Jacobian $J\psi_{1,2}$, we have $\det(J_x(\psi_{1,2} \circ \phi_1)) < 0$ (if no such x exists, then M is orientable). Then, since $U_1 \cap U_2$ is connected and $\det(J_z(\psi_{1,2} \circ \phi_1))$ is a continuous map of z in $U_1 \cap U_2$, we have $\det(J_z(\psi_{1,2} \circ \phi_1)) < 0$ for all $z \in U_1 \cap U_2$.

We consider the atlas

$$\{(U_1, \phi_1), (U_2, \psi \circ \phi_2)\}$$

which is another smooth atlas of M. Let $\psi'_{1,2}$ be the transition map of this atlas from $\phi_1(U_1 \cap U_2)$ to $\psi \circ \phi_2(U_1 \cap U_2)$ given by the formula

$$\psi'_{1,2}(z) = \psi \circ \phi_2 \circ \phi_1^{-1}(z)$$

for all $z \in \phi_1(U_1 \cap U_2)$. Then $\det(J_z \psi'_{1,2}) = \det(\psi) \det(J_z \psi_{1,2}) > 0$. This finishes the proof of Proposition 4.1.

Inspired by Proposition 4.1, we would like to consider the case of any finite atlas and try to find an analogous claim in this broad setting. Let M be a smooth manifold with a finite smooth atlas $\mathcal{U} = \{(U_1, V_1, \phi_1), \ldots, (U_m, V_m, \phi_m)\}$ such that for all $1 \leq i \leq j \leq m$, the intersection $U_i \cap U_j$ is connected (in particular, this intersection is allowed to be empty). We will call such atlases *proper* and associate a simple graph $\Gamma_{\mathcal{U}} = (V, E)$ by letting $V = \{v_1, \ldots, v_m\}$,

$$E = \{(v_i, v_j) : U_i \cap U_j \neq \emptyset, \psi_{ij} \text{ is orientation reversing}\}$$

where ψ_{ij} still denote the transition maps of the atlas $\mathcal{U}$. For example, in the case of an atlas as in Proposition 4.1, the associated graph will be isomorphic to K_2 - the complete graph on 2 vertices.

Theorem 4.2. *a) Let M be a smooth manifold with a proper atlas $\mathcal{U}$ and associated graph $\Gamma_{\mathcal{U}}$. Then M is orientable if and only if $\Gamma_{\mathcal{U}}$ is bipartite.*

b) For every finite simple graph Γ, there exists a 2-manifold M with a proper atlas $\mathcal{U}$ such that the associated graph $\Gamma_{\mathcal{U}}$ is isomorphic to Γ.

c) If Γ is a finite simple bi-partite graph, then there exists a closed 2-manifold M with a proper atlas $\mathcal{U}$ such that the associated graph $\Gamma_{\mathcal{U}}$ is isomorphic to Γ.

In part b), we will construct the manifold M explicitly. Let us remark that if Γ is bi-partite, then, by part a), the corresponding 2-manifold is necessarily orientable. However, we will present an explicit example of M also in the bi-partite case.

Proof. a) For the "if part", let $\Gamma_{\mathcal{U}} = (V, E)$ with $V = V^{(0)} \sqcup V^{(1)}$ such that for all $k \in \{0,1\}$ no two vertices of $V^{(k)}$ are connected with an edge. Without loss of generality, we may assume that $V^{(0)} = \{1, \ldots, r\}$ and $V^{(1)} = \{r+1, \ldots, m\}$. Let $n = \dim M$ and for all $1 \leq i \leq r$, let $\psi_i : V_i \to \mathbb{R}^n$ be an orientation reversing map. Then the new atlas

$$\begin{aligned}\mathcal{U}_1 &= \{(U_1, V_1, \psi_1 \circ \phi_1), \ldots, (U_r, V_r, \psi_r \circ \phi_r),\\ &\quad (U_{r+1}, V_{r+1}, \phi_{r+1}), \ldots, (U_m, V_m, \phi_m)\}\end{aligned}$$

will be oriented exactly as we demonstrated in the proof of Proposition 4.1.

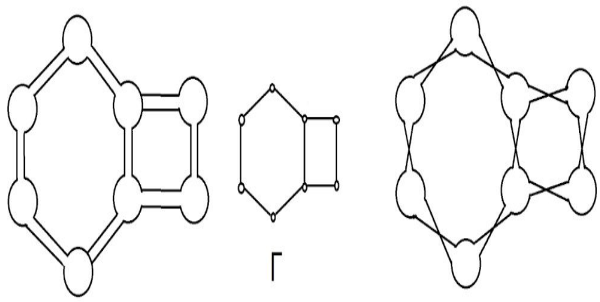
For the "only if part", suppose that M is orientable and the graph $\Gamma_{\mathcal{U}}$ is not bi-partite. Then it has an odd cycle; without loss of generality (by re-numerating vertices) we can assume that $(v_1, v_2, \ldots, v_s, v_1)$ is such an odd cycle where s is an odd number. M admits an oriented atlas

$$\mathcal{U}' = \{(U_1, V_1, \phi_1'), \ldots (U_m, V_m, \phi_m')\}$$

with the transition maps ψ_{ij}'. Then the maps $\psi_{1,2}', \psi_{2,3}', \ldots, \psi_{s-1,s}', \psi_{s,1}'$ are all orientation preserving, hence $\det J\psi_{j,j+1}' > 0$ for all $j \in \mathbb{Z}/s\mathbb{Z}$. We also have $\det J\psi_{j,j+1} < 0$ for all $j \in \mathbb{Z}/s\mathbb{Z}$. Let $\theta_i = \phi_i' \circ \phi_i^{-1}, 1 \leq i \leq s$. Then for all $j \in \mathbb{Z}/s\mathbb{Z}$, we have $\det J\theta_j > 0 \Rightarrow \det J\theta_{j+1} < 0$ and $\det J\theta_j < 0 \Rightarrow \det J\theta_{j+1} > 0$. But this is impossible because s is odd.

b) For the proof, we will need the notion of a *band sum* in a specific sense. A band, for us, will be just a rectangle $R = [0,1] \times [0,l], l > 0$ where we have marked the opposite sides $b_0 := \{0\} \times [0,l]$ and $b_1 := \{1\} \times [0,l]$. If D_0, D_1 are two disks with arcs C_0, C_1 at the boundaries $\partial D_0, \partial D_1$ respectively, then, by gluing b_0 with C_0 and b_1 with C_1 we obtain the band sum $D_0 \cup_b D_1$; the latter is just a topological disk again, but if we have a finite collection of disks and attach bands to some pairs of them, we might end up with a 2-manifold other than a disk. For the purposes of orientation, it is also useful to distinguish two types of disk bands: Assume that $D_0 \sqcup D_1$ is embedded in an oriented plane inheriting the orientation of it. The band can be oriented such that it agrees with this orientation - in this case, the band sum $D_0 \cup_b D_1$ will also be oriented, and we will call such a band sum *regular*. We can also imagine the band twisted so that the fixed orientaions of D_1 and D_2 cannot be extended to the one of the whole band sum $D_0 \cup_b D_1$; such a band sum will be called *twisted*. Topologically, this means that, when gluing, the orientation of b_0 agrees with the orientation of C_0 whereas the orientation of b_1 disagrees with the orientation of C_1 (or the opposite, the orientation of b_0 disagrees with the orientation of C_0 and the orientation of b_1 agrees with the orientation of C_1).

First, we will treat the case where $\Gamma = (V, E)$ is bi-partite, by constructing an orientable 2- manifold in this case. Let $V = \{v_1, \ldots, v_m\}$. Then we let $D_1, \ldots, D_m$ be disjoint oriented copies of $\mathbb{D}^2$. For each edge (e_i, e_j) we connect D_i and D_j with a regular band $b_{i,j}$ such that the bands $\{b_{i,j} : (v_i, v_j) \in E\}$ are mutually disjoint. The resulting manifold is an orientable surface with boundary (we can delete the boundary to obtain a manifold without a boundary). Then for all $i \in \{1, \ldots, m\}$, we let U_i be the open set consisting of the interior of the union of D_i together with all the bands attached to it. We orient each $U_i, 1 \leq i \leq m$ such that if (e_i, e_j) is an edge, then U_i and U_j are oriented in the opposite ways. Notice that the intersection $U_i \cap U_j$ will consist of the band b_{ij}, hence our atlas will be proper.



Now, let $\Gamma = (V, E)$ be any finite simple graph and $V = \{v_1, \ldots, v_m\}$. Let again $D_1, \ldots, D_m$ be oriented disjoint copies of $\mathbb{D}^2$. For each edge (e_i, e_j) we connect D_i and D_j with a twisted band $b_{i,j}$ such that the bands $\{b_{i,j} : (v_i, v_j) \in E\}$ are mutually disjoint. The resulting manifold will be a surface with boundary (and we can delete the boundary again). Now, for all $i \in \{1, \ldots, m\}$, we let U_i be the open set consisting of the interior of the union of D_i with all the bands attached to it. We orient all $U_i, 1 \leq i \leq m$ in the same way. Then if $(v_i, v_j) \in E$, then, since the band b_{ij} in between D_i and D_j is twisted, the transition map ψ_{ij} will be orientation reversing.

c) For the proof, we will use the following simple lemma.

Lemma 4.3. *Let $n \geq 4$ be an even number and $\mathcal{P} = A_0 A_1 \ldots A_{n-1}$ be an n-gone in the plane viewed as a topological subspace of the plane. Then there exist connected open subsets $W_i, i \in \mathbb{Z}/n\mathbb{Z}$*

of $\mathcal{P}$ such that the following conditions hold:

(i) for all $i \in \mathbb{Z}/n\mathbb{Z}, A_i \in W_i$;

(ii) for all $i, j \in \mathbb{Z}/n\mathbb{Z}$, if $i-j \in \{3, 5, \dots, n-3\}$, then $W_i \cap W_j = \emptyset$.

Proof. We may assume that $\mathcal{P}$ is a regular n-gon with side 1. Let $n = 2s$. Since the claim is obvious for $n = 4$, we may assume that $n \geq 6$ (so $s \geq 3$). We will consider two cases:

Case 1. s is even.

In this case, letting $s = 2k$, we can take open stripes $S_1, \dots, S_k$ around the diagonals

$$A_0A_s, A_2A_{s+2}, \dots, A_{s-2}A_{2s-2}$$

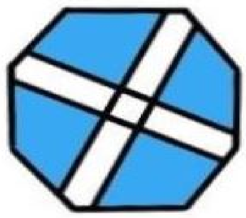
respectively, with stripes being geometrically symmetric with respect to the corresponding diagonals, such that the complement $\mathcal{P}\backslash \bigcup_{1\leq j\leq k} S_i$ consists of mutually disjoint quadrilaterals $Q_1, \dots Q_s$ such that $A_{2j-1} \in Q_j, 1 \leq j \leq s$ (see the octagon figure on the right in the case of $m = 8$). Then for sufficiently small $\epsilon > 0$, we may let

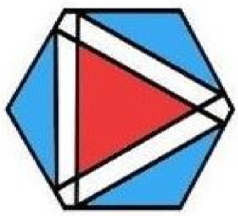
$$W_{2j-1} = \{x \in \mathcal{P} \mid d(x, Q_j) < \epsilon\}, 1 \leq j \leq s$$

and

$$W_{2j} = \{x \in S_{2j} \mid d(x, A_{2j}) < d(x, A_{2j+s}) + \epsilon\}, 0 \leq j \leq s-1$$

where d denotes the usual Euclidean distance (from a point to a set, as the infimum of all possible distances).





Case 2. s is odd.

In this case, we can take open stripes $S'_1, \dots, S'_s$ around the diagonals $A_0A_2, A_2A_4, \dots, A_{2s-2}A_0$ respectively such that the complement $\mathcal{P}\backslash \bigcup_{1\leq j\leq s} S_j$ consists of mutually disjoint triangles $T_1, \dots T_s$ and a central regular s-gon Δ such that $A_{2j-1} \in T_j, 1 \leq j \leq s$ (see the hexagon figure on the left in the case of $m = 6$). Then for sufficiently small $\epsilon > 0$, we may let

$$W_{2j-1} = \{x \in \mathcal{P} \mid d(x, T_j) < \epsilon\}, 1 \leq j \leq s$$

and

$$W_{2j} = \{x \in \mathcal{P} \mid d(x, (S'_{j+1}\backslash S'_{j+2}) \cup \Delta) < \epsilon\}, 0 \leq j \leq s-1$$

where we identify S'_{s+1} with S'_1. □

Now, we recall that in the proof of part b), in the orientable case, we obtain a compact 2-manifold M with boundary. The boundary of M consists of finitely many circles $C_1, \dots C_l$. For each $C_i, 1 \leq i \leq l$, let $U_{i_1}, \dots, U_{i_n}$ be the open sets constructed in the proof of part b), having an arc on C_i. We attach a disk to M by gluing the boundary to C_i and using Lemma 4.3 extend the sets $U_{i_1}, \dots, U_{i_n}$ to cover this disk so that the atlas still remains proper and the associated graph does not change (remains as Γ). □

Returning to the atlas that we had chosen on $\mathbb{RP}^n$ in Section 3, let us remark that the atlas $\mathcal{U} = \{(U_0, \phi_0), (U_1, \phi_1), \dots, (U_n, \phi_n)\}$ chosen on $\mathbb{RP}^n$ there is not proper, neither for odd nor for even n, because for all $0 \leq i < j \leq n$, the intersection $U_i \cap U_j$ will be disconnected.

Remark 4.4. In part b) of the theorem, we cannot claim the existence of a *closed* manifold. Indeed, let $\Gamma = K_n$, the complete graph on $n \geq 3$ vertices. If M is a closed 2-manifold with a finite proper atlas $\mathcal{U}$ and $\Gamma_{\mathcal{U}} \cong K_n$, then, since M has the topological dimension two, some three of open charts of $\mathcal{U}$ will have a common point. But this leads to an obvious contradiction.

Remark 4.5. Proposition 4.1 implies that non-orientable closed surfaces do not admit a proper atlas with two charts. In the case of an orientable closed surface, let us first show that they admit a natural atlas consisting of two charts. Indeed, for a genus g-surface $\Sigma_g \cong g\mathbb{T}^2$, we can use one of its standard embeddings $\Sigma \subset \mathbb{R}^3$ satisfying the following condition: For the xy-plane $\Pi = \{(x, y, z) : z = 0\}$, Σ is symmetric with respect to Π and the intersection $\Sigma \cap \Pi$ consists of $g+1$ circles $S_0, S_1, \dots, S_g$ such that $S_1, \dots, S_g$ are all contained within S_0, but no one of them is contained within the another one. Then, we can take the "northern and southern hemishperes" by letting $U_+ = \{(x, y, z) \in \Sigma : z > -\epsilon\}$ and $U_- = \{(x, y, z) \in \Sigma : z < \epsilon\}$; for sufficiently small ϵ, we will have $U_+ \cong \mathbb{R}^2 \cong U_-$. The intersection $U_+ \cap U_-$ will be a disjoint union of narrow stripes around the circles $S_i, 0 \leq i \leq g$, so this intersection will not be connected. Hence, the atlas formed of U_+ and U_- will not be proper. Indeed, one can show that Σ_g does not admit a proper atlas with two charts whenever $g \geq 1$. For any such atlas with open sets U_1, U_2, since the intersection $U_1 \cap U_2$ is connected, it contains a separating circle C with the complement Σ_g/C having two connected components M_1, M_2 each being homeomorphic to a punctured surface such that $M_1 \subseteq U_1, M_2 \subseteq U_2$. Since $g \geq 1$, one of these surfaces will have a positive genus, hence it will not be homeomorphic to an open subset of $\mathbb{R}^2$ (the one-point compactification of a punctured genus-g surface is homeomorphic to a genus-g surface, whereas the one-point compactification of an open subset of $\mathbb{R}^2$ is either homeomorphic to $\mathbb{S}^2$ or it is not a manifold).

5 Orientability of boundary and orientable manifolds as boundary

We already emphasized in the introduction that 1-manifolds without a boundary embedded in a plane bound a domain in it; any embedding of a circle bounds a disk, whereas an embedding of a line bounds two unbounded domains each homeomorphic to a plane. This fact can indeed be used to orient the circle and the line.

Proposition 5.1. *Let M be a smooth n-manifold embedded in $\mathbb{R}^{n+1}$. Suppose that M bounds a domain D in $\mathbb{R}^{n+1}$. Then M is orientable.*

For the proof of the above proposition, let us fix the positive orientation of $\mathbb{R}^{n+1}$, given by the choice of the standard basis $(e_1, \ldots, e_n, e_{n+1})$. Since $M = \partial\overline{D}$ and M are smooth, at every point $x \in M$, we can choose a unit vector $v(x)$ perpendicular to T_xM oriented "inside", i.e. towards D. Then we let $(u_1(x), \ldots, u_n(x))$ be a basis of T_xM such that $(v(x), u_1(x), \ldots, u_n(x))$ gives a positive orientation of $\mathbb{R}^{n+1}$. Then the n-tuple of vectors $(u_1(x), \ldots, u_n(x))$, as x varies over M, defines an orientation for M.

Proposition 5.1 is indeed very useful; for example, it can be used to show that the real projective plane and the Klein bottle are not (smoothly) embeddable in $\mathbb{R}^3$. This immediately follows by combining the claim of the proposition with the Jordan-Brouwer Separation Theorem, which states that *a closed smooth n-dimensional submanifold of $\mathbb{R}^{n+1}$ bounds a domain in it* and its complement has two connected components, one bounded and another one unbounded.

Proposition 5.1 can be generalized by replacing D with an arbitrary orientable manifold, in fact, by a similar proof.

Theorem 5.2. *Let N be a smooth manifold with boundary such that $M = \partial N$. If N is orientable, then M is also orientable.*

The proof is indeed very similar to the proof of Proposition 5.1. Let us fix an orientation ω of N. For all $x \in M$, let $v(x) \in T_xN$ be the unit tangent vector normal to M. Then $(u_1(x), \ldots, u_n(x))$ be a basis of T_xM such that $(v(x), u_1(x), \ldots, u_n(x))$ gives us the orientation ω. Then the vectors $(u_1(x), \ldots, u_n(x))$, as x varies over M, define an orientation for M.

Let us emphasize that the claim of Theorem 5.2 is not true without the orientability condition imposed on our manifold N. For any closed manifold M, the product $M \times [0, 1]$ has the boundary consisting of two copies of M, so, if we choose M to be non-orientable, we will obtain $M \sqcup M$ as the boundary of $M \times [0, 1]$ but $M \sqcup M$ is non-orientable. It is more interesting to ask if a connected non-orientable manifold can be a boundary, and the answer to this is again "yes". Take a quotient of $N = \mathbb{D}^2 \times [0, 1]$ by identifying $(x, 0)$ with $(-x, 1)$ for all $x \in \mathbb{D}^2$. Then we obtain a 3-manifold whose boundary is the Klein bottle. Thus, unlike $\mathbb{RP}^2$, the Klein bottle can bound a 3-manifold.

Being a boundary for a manifold is equivalent to having all Stiefel-Whitney numbers zero, by Thom's Theorem. [4] It is easy to compute Stiefel-Whitney numbers for the real projective plane and for the Klein bottle. For 2-manifolds, we have only two Stiefel-Whitney numbers: w_2 and w_1^2. The number w_2 as the top class vanishes in the non-orientable case. w_1 is not zero for both Klein bottle and the $\mathbb{RP}^2$. However, its square vanishes in the case of Klein bottle (but not in the case of $\mathbb{RP}^2$). Thus, for the Klein bottle, all Stiefel-Whitney numbers vanish.

The Stiefel-Whitney numbers of $\mathbb{RP}^n$ are all zero iff n is odd. Without computing these numbers, we can still show that $\mathbb{RP}^n$ is a boundary for odd n: Any manifold M admitting a free involution becomes a boundary. Indeed, let $f : M \to M$ be such an involution. Then the quotient $M \times [-1, 1]/\sim$ by the equivalence relation $(x, -1) \sim (f(x), -1), x \in M$ is a manifold whose boundary is homeomorphic to M. The manifold $\mathbb{RP}^n$ does admit a free involution for odd n; the standard $\mathbb{Z}/4\mathbb{Z}$ action on $\mathbb{S}^n$, in the quotient, produces a $\mathbb{Z}/2\mathbb{Z}$ action on $\mathbb{RP}^n$ without a fixed point.

It turns out that in small dimensions, orientability indeed helps a manifold to become a boundary. We know that the orientable cobordism group Ω_n is trivial for $n = 1, 2, 3$. This means that for $n \leq 3$, a closed orientable n-manifold is a boundary of an $(n + 1)$-manifold.

[4] More precisely, it follows by easier arguments that if a manifold is a boundary, then its Stiefel-Whitney numbers are all zero. Thom's Theorem proves the converse, which is much harder. For basic knowledge of Stiefel-Whitney numbers and classes, we refer the reader to [16].

6 Complex manifolds

It is interesting that some structures that we can impose on manifolds may indirectly (or not in an obvious way) imply orientability. A complex structure or a Lie group structure are amongst these. For the latter, it is quick to establish orientability: For a given Lie group G with $\dim G = n$, by choosing a basis $(v_1, \ldots, v_n)$ of the tangent space T_eG as an orientation of T_eG (here, e is the identity element), for any g, the map $\Phi_g : G \to G$ defined as $\Phi_g(x) = gx$ for all $x \in G$, is diffeomorphism sending e to g. Then we obtain a differential map $D_e\Phi : T_eG \to T_gG$ and let $(\Phi_g(v_1), \ldots, \Phi_g(v_n))$ be the basis defining the orientation of T_gG. Thus, we are choosing a frame on T_gG that changes smoothly as g runs in G. This defines an orientation locally on G that agrees on overlapping local charts. Hence, we produce an orientation for G. For complex manifolds though, establishing an orientation is not as straightforward. A smooth manifold M is called a complex manifold if the following two conditions hold:

1) $\dim M = 2n$ for some positive integer n, i.e. the dimension of our manifold is an even number;

2) The transition maps of our atlas are holomorphic maps from a domain of $\mathbb{C}^n$ to $\mathbb{C}^n$.

Holomorphicity of a map has strong analytical implications which we explore in complex analysis. It turns out that the implications of this carry over to topology and geometry of complex manifolds as well. In this section, our goal is to show that manifolds with a complex structure are orientable. In fact, complex manifolds are oriented - the complex structure already determines the orientation!

If the complex dimension of our complex manifold M is 1 (i.e. the real dimension is two) then the transition map is if the form $F : D_1 \to D_2$ where $D_1, D_2 \subseteq \mathbb{C}$ are domains and

$$F(z) = u(x, y) + v(x, y)\mathbf{i}$$

is a holomorphic map with $u(x, y), v(x, y)$ being the real and imaginary parts of $f(z) = f(x + y\mathbf{i})$ respectively. By the Cauchy-Riemann condition, $u_x = v_y$ and $u_y = -v_x$; then, for the Jacobian, we have

$$\det J_zF = \det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = u_x^2 + u_y^2 \geq 0.$$

If $u_x(z) = u_y(z) = 0$ at some point $z \in D_1$, then $u_y(z) = v_x(z) = 0$. But since F is a diffeomorphism, we cannot have $u_x(z) = u_y(z) = v_x(z) = v_y(z) = 0$. Hence $detJ_zF > 0$.

In an arbitrary complex dimension n, our transition map will be a map $F : D \to \mathbb{C}^n$ where D is a domain of $\mathbb{C}^n$ and $F = (F_1, \ldots, F_n)$ is given as n-tuple of holomorphic maps $F_j = u_j + v_j\mathbf{i}, 1 \leq j \leq n$. We again invoke the Cauchy-Rieamann conditions:

$$\frac{\partial u_j}{\partial x_k} = \frac{\partial v_j}{\partial y_k}, \quad \frac{\partial u_j}{\partial y_k} = -\frac{\partial v_j}{\partial x_k}, \; 1 \leq i, k \leq n$$

and obtain the Jacobian in the form

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$$

where $A = (a_{jk})_{1 \leq j,k \leq n}, B = (a_{jk})_{1 \leq j,k \leq n}$ with $a_{jk} = \frac{\partial u_j}{\partial x_k}$ and $b_{jk} = \frac{\partial u_j}{\partial y_k}$.

Performing, first, column operations and then row operations, we obtain that

$$\begin{aligned} \det \begin{bmatrix} A & B \\ -B & A \end{bmatrix} &= \det \begin{bmatrix} A + B\mathbf{i} & B \\ -B + A\mathbf{i} & A \end{bmatrix} \\ &= \det \begin{bmatrix} A + B\mathbf{i} & B \\ 0 & A - B\mathbf{i} \end{bmatrix} \\ &= \det(A + B\mathbf{i}) \det(A - B\mathbf{i}) \\ &= \det(A + B\mathbf{i}) \overline{\det(A + B\mathbf{i})} \\ &= |\det(A + B\mathbf{i})|^2 > 0. \end{aligned}$$

Thus, we proved that our complex manifold M is oriented.

7 Symplectic manifolds

A symplectic manifold is a smooth manifold equipped with a closed non-degenerate differential 2-form ω. Thus, for all $p \in M$, we have a skew-symmetric bilinear form $\omega(X, Y), X, Y \in T_pM$. Non-degeneracy means that for all $X \in T_pM$, if $\omega(X, Y) = 0$ for all $Y \in T_pM$, then $X = \mathbf{0}$. Closedness means that $d\omega = 0$ where d is the exterior derivative of the form. [5]

By choosing a basis $(X_1, \ldots, X_n)$ of T_pM (thus $n = dimM$), we can associate a skew-symmetric matrix $\Omega = (\omega(X_i, X_j))_{1 \leq i,j \leq n}$ to the 2-form ω. Then, non-degeneracy of ω is equivalent to non-singularity of the matrix Ω. In odd dimensions, any skew-symmetric matrix is singular; indeed, if A is a skew-symmetric $n \times n$ matrix for an odd number n, then

$$\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A) = -\det(A)$$

thus $\det(A) = 0$. So having a non-degenerate 2-form on a manifold forces the dimension n to be an even number.

[5] For basic elements of the theory of differential forms, we refer the reader to [24].

Let $n = 2m$. For $m = 1$ (i.e. $n = 2$), we can already orient our manifold by letting the basis (X, Y) of T_pM be positively oriented if $\omega(X, Y) > 0$ (the non-degeneracy implies that $\omega(X, Y) \neq 0$). Thus, all 2-dimensional symplectic manifolds are orientable and even oriented as the orientation is just defined by the form ω. But what to do in higher even dimensions? We need to try to use the non-degenerate skew-symmetric form ω again to somehow produce an orientation for any basis $(X_1, \ldots, X_n)$ of T_pM for $p \in M$ in a consistent way. In order to do this, we will choose a very creative (but now very standard and basic in the theory of symplectic manifolds) and conceptually meaningful method.

To establish orientability, we will not use the closedness of the form ω. Thus we will show that any manifold equipped with a non-degenerate 2-form is oriented. Our idea is to use the form $\eta := \omega^m = \omega \wedge \cdots \wedge \omega$. Since ω is a 2-form, $\eta = \omega^m$ will be a $2n$-form. Can we hope that for any basis $(X_1, \ldots X_n)$ of T_pM, the value $\eta(X_1, \ldots, X_n)$ is never zero?! If this is the case, then the value is either positive or (if we permute the first two vectors) negative, and we would orient the basis $(X_1, \ldots X_n)$ positively if $\eta(X_1, \ldots X_n) > 0$. It turns out that this dream of us is indeed a reality!

We already emphasized that the closedness of ω will not be used. In fact, we can work on each tangent space T_pM individually. We know that $\dim T_pM = 2n$ is even and ω is a non-degenerate skew-symmetric bilinear form on $V := T_pM$. Then V admits a basis

$$(e_1, \ldots, e_n, f_1, \ldots, f_n)$$

such that $\omega(e_i, e_j) = \omega(f_i, f_j) = 0, \omega(e_i, f_j) = \delta_i^j$ for all $1 \leq i, j \leq n$. Then ω can be written as $\sum_{i=1}^{n} e_i^* \wedge f_i^*$. Inductively on k, we compute that

$$\omega^k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} e_{i_1}^* \wedge f_{i_1}^* \wedge \cdots \wedge e_{i_k}^* \wedge f_{i_k}^*.$$

Letting $k = n$, we obtain that

$$\omega^n = n! \; e_1^* \wedge f_1^* \wedge \cdots \wedge e_n^* \wedge f_n^*.$$

Then $\omega^n(e_1, f_1, \ldots, e_n, f_n) = n!$ hence

$$\omega^n(e_1, f_1, \ldots, e_n, f_n) \neq 0$$

But ω^n is a bilinear skew-symmetric $2n$-form on a $2n$-dimensional vector space V, therefore, if it is non-zero for one basis, it will be non-zero for any other basis (just the sign of the value may change). Thus, we see that our dream from the previous paragraph is becoming true! In essence, in our argument in this section, we have used another characterization for orientability, namely, *a smooth n-manifold is orientable if and only if it admits a volume form, i.e. a nowhere-vanishing smooth n-form.*

8 Connected sum

The connected sum is defined for any two manifolds (and more generally, for manifolds with boundary) of the same dimension $n \geq 1$. We remove a ball in each of the manifolds and then glue them along a homeomorphism of the boundary sphere $\mathbb{S}^{n-1}$. As long as the manifolds are connected, the homeomorpshims (or diffeomorphism) type of the connected sum does not depend on the choice of the deleted balls and the gluing homeomorphism (diffeomorhism). The connected sum seems like an innocent operation on manifolds, but it may indeed spoil the structures. The symplectic structure passes through it; a connected sum of two symplectic manifolds is indeed symplectic! However, the complex structure fails! For example, $\mathbb{CP}^2 \# \mathbb{CP}^2$ does not admit a complex structure. The symplectic structure in general should be considered more flexible than the complex structure (for example, every finitely presented group is a fundamental group of a symplectic 4-manifold [8], but fundamental groups of complex 4-manifolds by far do not capture all finitely presented groups) but numerous other topological properties also fail to stay invariant under the connected sum operation.

We saw that complex manifolds and symplectic manifolds are not only orientable, but even oriented; the imposed structures (complex or symplectic) already define an orientation on the manifold. The relation of connected sum with orientability is different: *A connected sum of orientable manifolds is orientable but not necessarily oriented.* The difference-making factor here is that the gluing homeomorphism between the boundary spheres may or may not preserve the orientation (of these spheres).

The issue is similar to the gluing of any two manifolds with boundary along the boundary components. If M and N are manifolds with a boundary of the same dimension, and M_1, N_1 are homeomorphic connected components of the boundaries ∂M and ∂N respectively, then gluing M and N along a homeomorphism $M_1 \cong N_1$, will result in a manifold or manifold with boundary, which is orientable but not necessarily oriented. Orientedness will depend on whether the gluing homeomorphism respects orientation.

For orientability of a connected sum, the orientability of each of the summands is not only sufficient, but also necessary! For example, if we attach a Möbius band to a surface, it becomes non-orientable, and it will stay non-orientable if we keep attaching more handles or Möbius bands to it. As another application, we know that pairwise gluing of the sides of a $2n$-gon results in a closed 2-manifold. But which one? In the Figure 1 above, the sides of a 10-gon are glued pairwise to obtain a surface S_1. One can easily compute the Euler characteristic and see that $\chi(S_1) = 3-5+1 = -1$. For orientable closed surfaces we have $\chi(g\mathbb{T}^2) = 2-2g$ and for non-orientable surfases, $\chi(k\mathbb{RP}^2) = 2-k$; thus if the Euler characteristic is odd, then our closed surface is necessarily non-orientable; $\chi(S) = -1$ yields $2-k = -1$, hence $k = 3$. Thus $S_1 \cong 3\mathbb{RP}^2$. But in Fugure 2, sides of a 12-gon are glued pairwise to obtain a surface S_2. Again, we can easily compute the Euler characteristic $\chi(S_2) = 1-6+1 = -4$. This time the Euler characteristic is even and we have two candidates for the obtained surface: $3\mathbb{T}^2$ and $6\mathbb{RP}^2$.

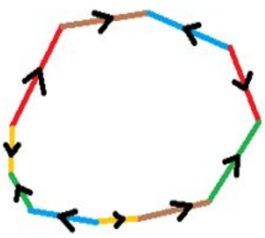


Figure 1

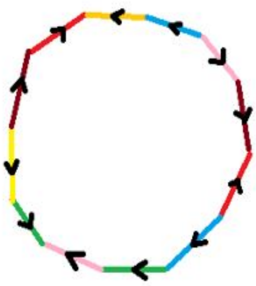


Figure 2

If in the gluing pattern of the sides of our polygon, we have two consecutive glued edges of the same orientation, then our surface is obtained by attaching a Möbius band to another surface, thus it is non-orientable. Even if the identified edges are not consecutive but have the same orientation, by cut and paste we can get the situation of two consecutive edges of the same orientation. In the gluing pattern of 12-gon in Figure 2, the brown edges have the same orientation, thus our surface is non-orientable. Hence $S_2 \cong 6\mathbb{RP}^2$.

9 Tangent bundle

For a smooth manifold M of $\dim M = n$, we can associate a tangent space T_pM of dimension n at every point $p \in M$. Then the set $TM := \bigsqcup_{p\in M} T_pM$ admits a structure of a smooth manifold. This manifold, called the tangent bundle of TM, is an important tool in differential topology as it allows to define the derivative (differential) of a smooth map. But TM is also interesting as an object providing an insight to the topology of the underlying manifold M.

The tangent bundle TM is homotopically equivalent to M, however, the relation of its homeomorphism type to M is rather intricate. If $TM \cong M \times \mathbb{R}^n$, then we call the manifold M parallelizable. In this case M is also combable (admits a nonvanishing continuous tangent vector field). For compact manifolds, the combability is equivalent to vanishing of the Euler characteristic, whereas parallelizability is a much more rare property. For example, the sphere $\mathbb{S}^n$ is parallelizable only for $n = 1, 3, 7$ (and combable for all odd n). The most notable class of parallelizable manifolds are Lie groups; the example $\mathbb{S}^7$ shows that the class of parallelizable manifolds is strictly larger.

In this section, we aim to show that the tangent bundle TM is orientable (and even oriented) even if the manifold M is not. Let M be a smooth manifold of dimension n and $\mathcal{U} = \{U_\alpha, \phi_\alpha\}_{\alpha\in I}$ be a smooth atlas on M with $V_\alpha = \phi_\alpha(U_\alpha), \alpha \in I$ such that V_α is an open set in $\mathbb{R}^n$, and $TV_\alpha = V_\alpha \times \mathbb{R}^n$. Hence, the differential map $D_*\phi_\alpha : TU_\alpha \to TV_\alpha$ obtains the form $D_*\phi_\alpha : TU_\alpha \to V_\alpha \times \mathbb{R}^n$.

The family $\{TU_\alpha\}_{\alpha\in I}$ forms a covering of M and $\{TU_\alpha, D_*\phi_\alpha\}_{\alpha\in I}$ becomes a smooth atlas. The transition maps

$$\Psi_{\alpha\beta} : (D_*\phi_\alpha) \circ (D_*\phi_\beta)^{-1} : V_\beta \times \mathbb{R}^n \to V_\alpha \times \mathbb{R}^n$$

will have a Jacobian of the form

$$J_{p,v} = \begin{bmatrix} A & 0 \\ B & A \end{bmatrix}$$

at the point $D_*(p, v)$ with $(p, v) \in TM, p \in M, v \in T_pM$, where A is the Jacobian of the transition map $\psi_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ of the original atlas $\mathcal{U}$ on M at the point $\phi_\beta(p)$. Then

$$\det(J_{p,v}) = \det \begin{bmatrix} A & 0 \\ B & A \end{bmatrix} = \det(A)^2 > 0$$

thus the transition maps $\Psi_{\alpha\beta}$ are orientation preserving. Hence, the atlas $\{TU_\alpha, D_*\phi_\alpha\}_{\alpha\in I}$ is oriented.

As emphasized and shown above, the tangent bundle TM is orientable even if M is not. But if the base manifold M itself is orientable, then recalling that the bundle map $\pi : TM \to M$ is locally trivial, for each $p \in M$, we can orient the tangent space T_pM in such a way that for some open neighborhood U of p the trivialization map

$TU \to U \times \mathbb{R}^m$ is *fiberwise orientation preserving*. This leads to an orientation of the entire TM, where we have built the orientation fiberwise; more precisely, we obtain the orientation of TM as a combination of orientations on the base M and on the tangent spaces $T_pM, p \in M$. The existence of an orientation built in this fashion is a subtler property called *bundle orientability.* If the base manifold is not orientable, then the bundle orientability is impossible, so we can in fact claim that *the tangent bundle TM is orientable as a bundle if and only if the base manifold M is orientable.*

The notion of bundle orientability can be extended to *fiber bundles* $F \to E \to M$, where, along with base M, the fiber F and the bundle E are also smooth manifolds. For manifolds with a bundle structure, it is interesting to investigate how, in the presence of such a structure, the orientability of the bundle E as a manifold or as a bundle is related to the orientability of the base M and the fiber F (when studying orientability of E as a manifold, we will refer to E as *the total space*). Observe that by local triviality of fiber bundles (see [3] or [21] for a definition of a fiber bundle), if the fiber F is non-orientable, then so is the total space E since the latter contains $F \times U$ as a submanifold where U is an open subset of $\mathbb{R}^m$ with $m = \dim M$. (Here, we use the fact that the product of two manifolds is orientable if and only if both factors are so. The "'if part" of this claim is straightforward, but the "only if part" is already a meaningful and recommended exercise on orientability of fiber bundles despite the fact that the product of manifolds is a trivial fiber bundle). On the other hand, non-orientability of the base M does not necessarily imply non-orientabiity of the total space E as the example $\mathbb{S}^1 \to \mathbb{RP}^3 \# \mathbb{RP}^3 \to \mathbb{RP}^2$ (1) shows (we leave it to the reader as a challenging exercise to find the maps of this fibration).

If we assume orientability of the base and fiber, the case of a $E =$ *Möbius band*, viewed as an $\mathbb{I}$-bundle over $\mathbb{S}^1$ shows that these assumptions do not necessarily imply the orientability of the total space E even as a manifold. We observe the same phenomenon in another example $\mathbb{S}^1 \to$ Klein Bottle $\to \mathbb{S}^1$ (2). Notice that fiber bundles (1) and (2) are both circle bundles, so the fiber is even a group (a Lie group). However, we want to make a better (more meaningful) use of the Lie group structure of the fiber, which prompts us to consider the particularly interesting case of *principal bundles*, where, demanding finer properties, we want the fiber F to be a Lie group acting on E fiberwise, with the action being smooth, free and transitive on each fiber (see e.g. [3] for a precise definition). Being a principal bundle turns out to be rather restrictive. For example, none of the bundles (1) and (2) is principal. As a generalization of (1), let us note that circle bundles over closed surfaces form a rich class of 3-manifolds (called Seifert manifolds), but by far, not all of them are principal bundles.

For principal bundles, it is worth to return to the question that we asked earlier: If the base M is non-orientable, does this imply that the total space E is non-orientable? The answer is again "no" as seen from the example of $\mathbb{Z}/2\mathbb{Z} \to \mathbb{S}^2 \to \mathbb{RP}^2$. But then we can ask what if the fiber F is also connected? An interesting example of such arises as the space Rat_2 of quadratic rational maps studied in [15]. A quadratic rational map is defined as

$$f(z) = \frac{p(z)}{q(z)} = \frac{a_0z^2 + a_1z + a_2}{b_0z^2 + b_1z + b_2}$$

where $\max\{\deg(p), \deg(q)\} = 2$. Milnor shows that Rat_2 contains a 5-dimensional compact manifold M^5 as a deformation retract where M^5 is an $SO(3)$-bundle over $\mathbb{RP}^2$ (see Theorem 2.1 in [15]). The manifold M^5 is also non-orientable! This is a consequence of the fact that the fiber $SO(3)$ of our principal bundle is connected, therefore, non-orientability of the base M will result in the non-orientability of the total space E. Any orientation of the latter can be used to obtain an orientation of the base leading to a contradiction. Indeed, notice that the fiber $F = SO(3)$, being a Lie group, is orientable. Secondly, by the action of the fiber F as a Lie group, the fibers are glued so that there is no "twist" in the orientation. Hence, if E is orientable, we obtain that $M \times U$ is orientable for an open neighborhood U of identity in F. But this implies that M is orientable, which is a contradiction.

In the principal bundle $\mathbb{Z}/2\mathbb{Z} \to \mathbb{S}^2 \to \mathbb{RP}^2$, the fact that the fiber $\mathbb{Z}/2\mathbb{Z}$ is a discrete group of just two elements allows us "to fix" the non-orientability of the base $\mathbb{RP}^2$. It turns out this can be generalized to a broader tool for any non-orientable manifold, as we will explore in the next section.

A particular example of a principal bundle, called the *frame bundle*, also allows us to redefine orientability and to view orientability as a geometric structure on smooth manifolds. Consider the frame bundle $\mathrm{Fr}M$ on a smooth n-manifold M as a principal bundle with the structure group $GL(n, \mathbb{R})$. In the argument at the beginning of Section 7, we have essentially used the fact *orientability of M is equivalent to the existence of a smooth section of the bundle map* $\mathrm{Fr}M \to M$. On the other hand, a *geometric structure* on M is a reduction $GL(n,\mathbb{R}) \to G$ of the structure group of the principal frame bundle from $GL(n,\mathbb{R})$ to a particular Lie subgroup $G \leq GL(n,\mathbb{R})$. Examples of geometric structures are an almost complex structure, a Euclidean structure, an almost symplectic structure, etc. where the corresponding Lie subgroup G is $GL(\frac{n}{2},\mathbb{C}), O(n), Sp(n,\mathbb{R})$ respectively (for the complex and symplectic cases, we assume that n is even). In the case where $G \leq GL_+(n,\mathbb{R})$, the manifold is orientable.

10 Orientable double covers and 3-manifolds

For a smooth manifold M with a countable smooth atlas $\mathcal{U} = \{U_\alpha, \phi_\alpha\}_{\alpha\in I}$ on M, we can introduce another manifold $\tilde{M}$ as a 2-sheeted cover of M. The underlying set of $\tilde{M}$ can be identified with $M \times \{-1,1\}$, however, $\tilde{M}$ will not necessarily be diffeomorphic (or homeomorphic) to $M\times\{-1,1\}$ with the natural manifold structure on the latter as a disjoint union of two copies of M. In fact, we will have $\tilde{M} \cong M \times \{-1,1\}$ precisely in the case when M is orientable. If M is non-orientalbe, our manifold $\tilde{M}$ will be drastically different: it will be connected (if M itself is connected) and orientable!

An important idea in the construction of $\tilde{M}$ is, in fact, to realize its identification with $M \times \{-1,1\}$ in an indirect way. For each $\alpha \in I$ we also define the pair $\{U_\alpha, \phi'_\alpha\}$ such that ϕ'_α produces an orientation on U_α that is opposite to that of ϕ_α, i.e. the Jacobian $J(\phi'_\alpha \circ \phi_\alpha^{-1})$ has a negative determinant at each $x \in \phi_\alpha(U_\alpha)$.

We will assume that $\mathcal{U}$ forms a basis of M and for all $\alpha,\beta \in I$, the intersection $U_\alpha \cap U_\beta$ is connected. We consider the topological spaces $U_\alpha \times \{1\}, U_\alpha \times \{-1\}, \alpha \in I$. Each of these is a smooth submanifold of M by the standard embeddings $(x,1) \to x$ and $(x,-1) \to x$. Let

$$I^+_\alpha : U_\alpha \times \{1\} \to U_\alpha \text{ and } I^-_\alpha : U_\alpha \times \{-1\} \to U_\alpha$$

be these projections. We do the following gluing of these submanifolds: If $\alpha,\beta \in I$ such that $U_\alpha \cap U_\beta \neq \emptyset$, then we glue

1) $U_\alpha\times\{1\}$ and $U_\beta\times\{1\}$ if the Jacobian $J(\phi_\alpha\circ\phi_\beta^{-1})$ has a positive determinant;

2) $U_\alpha \times \{-1\}$ and $U_\beta \times \{-1\}$ if the Jacobian $J(\phi_\alpha \circ \phi_\beta^{-1})$ has a positive determinant;

3) $U_\alpha \times \{1\}$ and $U_\beta \times \{-1\}$ if the Jacobian $J(\phi_\alpha \circ \phi_\beta^{-1})$ has a negative determinant.

Each time, the gluing is along the intersection $U_\alpha \cap U_\beta$ which we have assumed to be connected. It is straightforward to check that after these gluings we obtain a second countable Hausdorff space locally homeomorphic to $\mathbb{R}^n$ taking the smooth structure from M.

For closed 2-manifolds, the non-orientable ones are of the form $k\mathbb{RP}^2$ for a positive integer k. The orientable double cover of it will be homeomorphic to $g\mathbb{T}^2$ where $g = k-1$. So, in particular, every closed orientable 2-manifold is a double cover of a non-orientable one. Notice that, in general, the Euler characteristic of a double cover is twice the Euler characteristic of the non-orientable base manifold, hence the Euler characteristic of a double cover is even. This shows that in higher dimensions, not every orientable manifold is a double cover; for example, $\mathbb{CP}^2$, being a complex manifold, is orientable, but it is not a double cover since $\chi(\mathbb{CP}^2) = 3$ is odd. However, notice that this Euler characteristic argument breaks down in the odd-dimensional case, as $\chi(M) = 0$ for all closed manifolds of odd dimension. But we can still find plenty of examples in any dimension > 1. In the particularly interesting case of 3-manifolds, the 3-sphere $\mathbb{S}^3$ does not cover any non-orientable manifold. This follows from Synge's Theorem, which states that *a closed odd-dimensional Riemannian manifold of positive sectional curvature is orientable* [22].

Establishing the existence of an orientable cover also implies that the universal cover of a manifold is necessarily orientable. More generally, a non-orientable manifold can never be simply connected; a loop witnessing non-orientability cannot be contracted to a point.

Let us emphasize that this does not necessarily create a torsion in the fundamental group as the example of a torsion-free group

$$\begin{aligned}\pi_1(\text{Klein Bottle}) &\cong \langle a,b \mid aba^{-1} = b^{-1}\rangle \\ &\cong \langle c,d \mid c^2 = d^2\rangle\end{aligned}$$

shows. However, at the homology level, it is indeed true that the first homology group $H_1(M)$ of a non-orientable manifold M contains a torsion.[6]

The existence of an orientable double cover implies that every non-orientable manifold has an *orientable partner*. This partner is unique up to a diffeomorphism. (Let us warn that a double cover of an *orientable* manifold may not be unique. For example, $\mathbb{RP}^3$ has two different double covers such as $\mathbb{S}^3$ and $\mathbb{RP}^3 \sqcup \mathbb{RP}^3$.) The converse is not true in general: an orientable manifold does not always have a non-orientable part-

[6] A closed, connected n-manifold M has a 2-torsion in the homology group $H_{n-1}(M)$ [9]. For $n = 3$, by Poincare Duality, we obtain a 2-torsion in $H_1(M)$.

ner. This prompts a question which we intentionally phrase in a vague form: *which manifolds are more, orientable or non-orientable?*

To address the above question, we will establish a broad class of orientable manifolds not covering a non-orientable manifold (so one can justify the thinking that there are a lot more orientable manifolds than the non-orientable ones). Recall that in the class of closed 2-manifolds, this does not happen. By classification, these manifold come in two series, $n\mathbb{T}^2, n \geq 0$ - the orientable ones, and $(n+1)\mathbb{RP}^2$ - the non-orientable ones, and for the same value of n, $n\mathbb{T}^2$ and $(n+1)\mathbb{RP}^2$ are partners. However, the world of 3-manifolds is rich with examples. The classification theorem for 3-manifolds is the now-established Geometrization Conjecture of Thurston. It can be stated as follows: *in every closed 3-manifold M, there exists a finite disjoint list of spheres and tori such that the complement of these in M is a disjoint union of 3 manifolds (with boundary) each supporting one of the following eight geometries* ([23], [19]) :

$$\mathbb{H}^3, \mathbb{E}^3, \mathbb{S}^3, \mathrm{Nil}, \mathrm{Solv}, \mathbb{H}^2 \times \mathbb{R}, \mathbb{S}^2 \times \mathbb{R}, \widetilde{SL(2,\mathbb{R})}.$$

In the following theorem, we collect and unify various facts scattered in the literature.

Theorem 10.1. *For each of the geometries*

$$\mathbb{H}^3, \mathbb{E}^3, \mathbb{S}^3, \mathrm{Nil}, \mathrm{Solv}, \mathbb{S}^2 \times \mathbb{R}, \widetilde{SL(2,\mathbb{R})},$$

there exists a closed orientable 3-manifold admitting that geometry and not covering a non-orientable manifold. Such examples do not exist in the geometry $\mathbb{H}^2 \times \mathbb{R}$

For the proof, we will need the following *well-known* fact which is interesting in its own right.

Proposition 10.2. *The fundamental group of a non-orientable closed 3-manifold is infinite.*

Proof. By Poincaré duality, the Euler characteristic of any odd dimensional closed manifold is zero. Let M be a non-orientable closed 3-manifold and $\pi_1(M)$ be finite. Then $H_1(M;\mathbb{Q}) = 0$ (by finiteness of π_1) and $H_3(M;\mathbb{Q}) = 0$ (by non-orientability of M). Then, letting b_i be the rank of $H_i(M;\mathbb{Q})$, we have

$$\begin{aligned} 0 = \chi(M) &= b_0 - b_1 + b_2 - b_3 \\ &= 1 - 0 + b_2 + 0 = 1 + b_2 \geq 1 \end{aligned}$$

which is a contradiction. □

The quantity b_i is called the *i-th Betti number.* The proposition also implies that for a non-orientable closed 3-manifold, the first Betti number is positive.

We also recall that a closed 3-manifold group always has a *balanced presentation*, i.e. a presentation with equal number of generators and relations. This implies that *any closed 3-manifold group has a non-negative deficiency.* Recall that for any finitely presented group G, the deficiency $\mathrm{def}(G)$ is defined as the maximum of the quantities $m(\mathcal{P}) - n(\mathcal{P})$ on all finite presentations $\mathcal{P}$, where $m(\mathcal{P}), n(\mathcal{P})$ denote the number of generators and the umber of relations of the presentation $\mathcal{P}$ respectively. Among the free Abelian groups $\mathbb{Z}^d, d \geq 1$, one can easily see that these groups have a balanced presentation if and only if $1 \leq d \leq 3$, so for $d \geq 4$, $\mathbb{Z}^d$ is not a closed 3-manifold group (the group $\mathbb{Z}^2$ is also not a closed 3-manifold group but for a more sophisticated reason). One can compute that $\mathrm{def}(\mathbb{Z}^d) = d - \binom{d}{2}$ and $\mathrm{def}(\mathbb{F}_d) = d$ for all $d \geq 1$.

We will collect two very useful facts in the following proposition.

Proposition 10.3. *a) (J.Howie, [11]) If a group G is non-trivial and finitely presented with deficiency 0 and H is finite and non-trivial, then* $\mathrm{def}(G \times H) < 0$.

b) (J.Hillman, [10]) If G is a finitely presented group with elementary amenable normal subgroup U whose finite subgroups have bounded order and which has no nontrivial finite normal subgroup, then $\mathrm{def}(G) \leq 1$. *If* $\mathrm{def}(G) = 1$ *and $G \ncong \mathbb{Z}$, then* c.d.$G = 2$ *and either $G \cong \mathbb{Z}^2$ or $U \cong \mathbb{Z}$.*

From 10.3 we immediately obtain the following:

Proposition 10.4. *Let M be a closed connected 3-manifold and $G = \mathbb{Z} \ltimes_\phi \mathbb{Z}^2$ for some automorphism $\phi \in \mathrm{Aut}((\mathbb{Z}^2)$. Then the fundamental group $\pi_1(M)$ cannot be isomorphic to $G \oplus (\mathbb{Z}/2\mathbb{Z})$.*

Proof. Let us recall that $\mathrm{Aut}(\mathbb{Z}^2) \cong GL(2,\mathbb{Z})$, so let $\mathbb{Z}$ act on $\mathbb{Z}^2$ by a matrix $A \in GL(2,\mathbb{Z})$. Considering the mapping torus on the 2-dimensional torus $\mathbb{T}^2$, we obtain that G is a closed 3-manifold group. Hence $\mathrm{def}(G) \geq 0$. On the other hand, by Proposition 10.3 b), we obtain that $\mathrm{def}(G) \leq 0$. Hence $\mathrm{def}(G) = 0$. Then Proposition 10.3 a) yields $\mathrm{def}(G \oplus (\mathbb{Z}/2\mathbb{Z})) < 0$. Hence $G \oplus (\mathbb{Z}/2\mathbb{Z})$ is not a closed 3-manifold group. □

We will present the proof of Theorem 10.1 as a case-by-case discussion:

$\mathbb{H}^3$-*geometry:* Let us recall that the Seifert-Weber dodecahedral space **SW** is a closed hyperbolic 3-manifold with a perfect fundamental group G.

Its universal cover is $\mathbb{H}^3$, thus, by the basic covering theory, **SW** is aspherical. If **SW** double-covers a non-orientable 3-manifold M, then M is also aspherical and G is an index two subgroup of $\pi_1(M)$. Then, since G is perfect, the Abelianization $\pi_1(M)/[\pi_1(M), \pi_1(M)]$ is finite. This contradicts the fact that the first Betti number is positive.

$\mathbb{E}^3$-*geometry:* Up to a homeomorphism (and up to a diffeomorphism) there are 10 flat (i.e. Euclidean) closed 3-manifolds [26]; four of them are non-orientable, but six of them orientable. Hence, there exist exactly two closed Euclidean 3-manifolds which do not double cover a non-orientable one.

$\mathbb{S}^3$-*geometry:* As we pointed out earlier, there are no closed non-orientable spherical 3-manifolds, therefore, for example, $\mathbb{S}^3$ or Lens spaces do not cover a non-orientable 3-manifold.

Nil-*geometry:* All compact 3-dimensional nilmanifolds are of the form $H_3(\mathbb{R})/\Gamma_r$ (see [2], page 25) where $H_3(\mathbb{R})$ is the 3-dimensional Heisenberg group (the group of 3×3 unipotent matrices over the reals), r is a non-zero integer and

$$\Gamma_r = \begin{bmatrix} 1 & a & \frac{c}{r} \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}.$$

All these manifolds are orientable, hence, in the three-dimensional nilgeometry, there are no closed non-orientable manifolds. Therefore, none of the orientable nilmanifolds $H_3(\mathbb{R})/\Gamma_r$ double cover a non-orientable one.

Solv-*geometry:* Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$. The matrix A, with $|trA| > 2$, induces a hyperbolic homeomorphism of $\mathbb{T}^2$. We consider the mapping torus M of this homeomorphism, which is an orientable solv-manifold with a fundamental group $G = C \ltimes K \cong \mathbb{Z} \ltimes_A \mathbb{Z}^2$ where $C \cong \mathbb{Z}$ and $K \cong \mathbb{Z}^2$. Assume that M double-covers a non-orientable manifold N and let $H = \pi_1(N)$. Then G is an index-two subgroup of H. By Proposition 10.4, H cannot be isomorphic to $G \oplus \mathbb{Z}^2$. On the other hand, A is not a square of any matrix in $SL(2, \mathbb{Z})$ because, being equal to $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ modulo 2, it is not a square in the finite quotient group $SL(2, \mathbb{Z}/2\mathbb{Z})$ (the latter is isomorphic to the permutation group S_3 on three symbols). Hence H cannot be of the form $C' \ltimes_A \mathbb{Z}^2$ where $C' = \langle t \rangle$ is an infinite cyclic group such that the square of the generator of the acting group C' (i.e. the element t^2) is equal to the standard generator of the acting group C in $G = C \ltimes K$. Then the only possibility is that $H \cong C \ltimes L$ where L contains K as an index-two subgroup. Then $L \cong \mathbb{Z}^2$ or $L = \pi_1(\text{Klein bottle}) = \langle a, b \mid aba^{-1} = b^{-1} \rangle$. In the latter case, K is the subgroup of L generated by a^2 and b. Then $tat^{-1} = a^m b^n$ for some $m, n \in \mathbb{Z}$. Then $ta^2t^{-1} = a^{2m} b^{n+(-1)^m n}$. Notice that the number $n + (-1)^m n$ is always even. But t acts as the matrix A on K, thus $ta^2t^{-1} = a^2b^3$. This contradiction shows that L is not isomorphic to $\pi_1(\text{Klein bottle})$. Hence $L \cong \mathbb{Z}^2$. But in this case, we have $\pi_1(N) \cong C \ltimes L \cong \mathbb{Z} \ltimes_A \mathbb{Z}^2$ and since $\det A = 1$, we obtain that the solvmanifold N is also orientable. Contradiction.

$\mathbb{H}^2 \times \mathbb{R}$-*geometry:* All closed, connected, orientable models of this geometry are circle bundles over surfaces, i.e. $\mathbb{S}^1$-bundles over Σ_g for $g \geq 2$. Since $\Sigma_g \cong g\mathbb{T}^2$ covers $(g+1)\mathbb{RP}^2$, we obtain that every closed orientable manifold in this geometry double-covers a non-orientable one.

$\mathbb{S}^2 \times \mathbb{R}$-*geometry:* In this geometry, there are exactly four closed manifolds [19]:

$$\mathbb{S}^2 \times \mathbb{S}^1, \mathbb{RP}^2 \times \mathbb{S}^1, \mathbb{RP}^3 \# \mathbb{RP}^3, MT_-(\mathbb{S}^2)$$

where $MT_-(\mathbb{S}^2)$ is the mapping torus of $\mathbb{S}^2$ with the orientation reversing isometry. The manifolds $\mathbb{S}^2 \times \mathbb{S}^1, \mathbb{RP}^3 \# \mathbb{RP}^3$ are orientable whereas the manifolds $\mathbb{RP}^2 \times \mathbb{S}^1, MT_-(\mathbb{S}^2)$ are non-orientable. Remarkably, all of these manifolds are double covered by the same manifold $\mathbb{S}^2 \times \mathbb{S}^1$. Hence, the orientable manifold $\mathbb{RP}^3 \# \mathbb{RP}^3$ cannot double cover either of $\mathbb{RP}^2 \times \mathbb{S}^1$ or $MT_-(\mathbb{S}^2)$

$\widetilde{SL(2, \mathbb{R})}$-*geometry:* In this geometry, all closed manifolds are orientable. Thus, none of them double covers a closed non-orientable manifold. Manifolds modeled on these geometries are Seifert fibered with non-trivial Seifert bundle. If a Seifert fibered manifold is non-orientable, then the Euler class of the Seifert bundle of the two-fold orientable cover is zero (since there is an orientation-reversing involution either reversing the orientation of the fibers or the base), so a non-orientable manifold cannot be modeled on one of these geometries.

Thus, we proved Theorem 10.1 by a case-by-case discussion. In the proof, we also observed that in three of the eight geometries (spherical, Nil, $\widetilde{SL(2, \mathbb{R})}$) there are no non-orientable models. We also would like to mention that the thinking *there are more orientable manifolds than the non-orientable ones* is also justified in [1] for $\mathbb{P}^2$-irreducible 3-manifolds of fixed complexity c for all values of $1 \leq c \leq 6$; in fact, non-orientable $\mathbb{P}^2$-irreducible 3-manifolds start emerging only at the complexity $c = 6$.

11 Fundamental groups of non-orientable manifolds

Let us now recall that any finitely presented group can be realized as the fundamental group of closed, smooth, and orientable 4-manifold [25, 13]. Roughly speaking, given a group G with finite presentation $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$, we can take m polygons corresponding to relators and glue them along the labels corresponding to the generators to construct a 2-complex K with $\pi_1(K) \cong G$, then embed the 2-complex in $\mathbb{R}^5$ and obtain the required manifold M^4 as the boundary of a tubular neighborhood of the image of the embedding. This manifold will be oriented by construction. It is therefore natural to ask: *Can one find a closed non-orientable manifold M with $\pi_1(M) \cong G$?* If such M exists, then the fundamental group of its orientable double cover injects into G as an index two subgroup. So, it is necessary that the given finitely presented group G has a subgroup of index two. It turns out that this condition is also sufficient as can be shown by the construction similar to the one in the orientable case. This motivates us to focus on the following broad question: *Which finitely presented groups have an index two subgroup?*

Clearly, finite simple groups and finite groups of odd order do not possess subgroups of index two. In the latter class, by Feit-Thompson Theorem, all groups are solvable. As another example, any group generated by elements of odd order does not possess a subgroup of index two.

We will investigate the question about index-two subgroups in the classes of nilpotent groups and hyperbolic groups.[7] Let us recall that every finitely generated nilpotent group is finitely presented (hence, nilpotent groups are coherent which is a property also shared by 3-manifold groups [20]). Hyperbolic groups are finitely generated by definition, and it is also known that they are finitely presented (cf. [7]). Hyperbolic groups are called *elementary* if they are virtually cyclic. For non-elementary word hyperbolic groups, we will mention the following important fact, which has many applications in combinatorial group theory.

Theorem 11.1. *(See [8], [7]) Let Γ be a non-elementary word hyperbolic group with a finite generating set X and the corresponding left - invariant Cayley metric $d(\cdot,\cdot)$. Then for all non-torsion element $\gamma \in \Gamma$ and for all $R > 0$, there exists a natural N such that for all natural $n \geq 1$, the quotient $\Gamma/\langle \gamma^{nN} = 1 \rangle$ is non-elementary word hyperbolic and the quotient map $\pi : \Gamma \to \Gamma/\langle \gamma^n = 1 \rangle$ is injective on the ball*

$$B_R(1) = \{g \in \Gamma : d(g, 1) \leq R\}$$

The above theorem is very useful (with many applications in combinatorial group theory), but there are some difficulties in applying it to our question mainly because we have to consider exponents $Nn, n \geq 1$ which may not be an odd number. This forces us to prepare a more complex setting and use a much more complicated result, namely Theorem 7.9 from [4].

Let us recall that a word hyperbolic group with a left-invariant Cayley metric is a hyperbolic space (in the sense of Gromov) and isometries of a hyperbolic space X can be classified as *hyperbolic* (elements with two fixed points on the boundary ∂X one of which is attractive and another one repelling), *parabolic* (elements with one fixed point on the boundary ∂X where attractive and repelling points coincide) and *elliptic* (all other non-identity isometries; equivalently, an elliptic isometry is a non-identity isometry such that the orbit of any point of X under the iterations of it is bounded). The Cayley graph of a word hyperbolic group Γ is special and enjoys the following properties: a non-torsion element of Γ acts as a hyperbolic element, and torsion elements are elliptic (so we have no parabolic elements). The boundary $\partial\Gamma$ is a compact metric space and every non-torsion element $\gamma \in \Gamma$ has two fixed points on the boundary $\partial\Gamma$, P_γ and Q_γ; P_γ and Q_γ are attractive and repelling points of γ respectively, i.e. for all open neighborhoods U, V of P_γ and Q_γ respectively, there exists $N \geq 1$ such that for all $n > N$, $\gamma^n(\partial\Gamma \backslash V) \subseteq U$ and $\gamma^{-n}(\partial\Gamma \backslash U) \subseteq V$. A torsion element may have no fixed point or many fixed points (finitely many or infinitely many fixed points); in particular, an elliptic element may fix all points of $\partial\Gamma$. But if N is the subgroup formed of elements of Γ fixing $\partial\Gamma$ pointwise, then N is a finite normal subgroup and Γ/N is also a word hyperbolic group such that no non-identity element of it fixes the boundary pointwise; moreover, if Γ is non-elementary, then so is Γ/N. The following proposition is borrowed from [12].

Proposition 11.2. *Let Γ be a non-elementary word hyperbolic group. Then $\partial\Gamma$ is infinite and the set $\{(P_\gamma, Q_\gamma) : \gamma \text{ is a non-torsion element}\}$ is dense in $\partial\Gamma \times \partial\Gamma$.*

We will prove the following result.

Theorem 11.3.

a) Every finitely generated infinite nilpotent group has a subgroup of index two.

[7] by hyperbolic groups, we mean *word hyperbolic groups* in the sense of Gromov.

b) Every non-elementary word hyperbolic group has a non-elementary word hyperbolic quotient without a subgroup of index two.

Proof. a) Let Γ be a finitely generated infinite nilpotent group. It is known that the torsion elements form a normal subgroup $Tor(\Gamma)$ and $\Gamma/Tor(\Gamma)$ is torsion free. Let $\pi_1 : \Gamma \to \Gamma/Tor(\Gamma)$ be this quotient map. The quotient

$$H := \Gamma/Tor(\Gamma)$$

as a finitely generated torsion-free nilpotent group embeds in $U_n(\mathbb{Z})$ - the group of $n \times n$ upper-triangular unipotent integral matrices - for some $n \geq 1$.

Let $G_i, i \geq 1$ be the lower central series of $U_n(\mathbb{Z})$. Then $G_1 = U_n(\mathbb{Z})$ and $G_i, 1 \leq i \leq n$ is the group of upper-triangular unipotent integral matrices such that the first $(i-1)$ diagonals are zero. Let j be the biggest integer such that $G_j \cap H$ is non-trivial. Then there exists an element $h \in H \backslash [H, H]$ which is represented by a unipotent matrix such that for all $1 \leq i \leq j-1$, the i-th diagonal is zero, but there exists a non-zero entry on the j-th diagonal. Then the quotient $H/[H, H]$ (which is finitely generated Abelian group) contains an infinite torsion-free Abelian group, hence it contains an index two subgroup L. Let $\pi_2 : H \to H/[H, H]$ be the Abelianization map and $\pi : \Gamma \to L$ be the composition $\pi = \pi_2 \circ \pi_1$. Then $\pi^{-1}(L)$ is an index two subgroup of Γ.

b) Let Γ be a non-elementary word hyperbolic group. We will fix a finite generating set $S = \{s_1, \dots, s_m\}$ with $1 \notin S$. If Γ has a finite normal subgroup N, then by taking the quotient Γ/N (and, by abuse of notation, redenoting it by Γ again) we can assume that Γ has no finite normal subgroup, so any non-identity element of Γ acts non-trivially on the boundary $\partial\Gamma$.

Let $R \in \partial\Gamma$ and $A = \{s_i(P) : 1 \leq i \leq m\} = \{R_1, \dots, R_k\}$. By Proposition 11.2, we can choose a hyperbolic element $\gamma \in \Gamma$ with attractive and repelling points $P := P_\gamma, Q := Q_\gamma \in \partial\Gamma \backslash A$ and open neighborhoods $U, V, W, U_i, 1 \leq i \leq k$ of P, Q, R, P_i respectively such that

$$U \cap V = \emptyset$$
$$(U \cup V) \cap (\bigcup_{1 \leq i \leq k} U_i) = \emptyset,$$
$$W \cap (V \cup U \cup \bigcup_{1 \leq i \leq k} U_i) = \emptyset$$

Then there exists a natural N such that for all natural $n \geq N$, $\gamma^n(\partial\Gamma \backslash V) \subseteq U$ (and, in particular, $\gamma^n(W) \subseteq U$) and $\gamma^{-n}(\partial\Gamma \backslash U) \subseteq V$. Then for all $1 \leq i \leq m$ and $n \geq N, q \geq 1$, we have $(s_i\gamma^n)^q(R) \in \partial\Gamma \backslash W$, so $s_i\gamma^n$ is not a torsion. Notice that the finite set

$$S'_n = \{\gamma, \gamma^n s_1, \gamma^{2n} s_2, \dots, \gamma^{mn} s_m\}$$

consisting of non-torsion elements also generates Γ. Moreover, for sufficiently large p, the elements $\gamma^p, (\gamma^n s_1)^p, \dots, (\gamma^{mn} s_m)^p$ generate a free group of rank $m+1$. Then for sufficiently large p, the group Γ is relatively hyperbolic with respect to the family

$$\{\langle\gamma^p\rangle, \langle(\gamma^n s_1)^p\rangle, \dots, \langle(\gamma^{mn} s_m)^p\rangle\}$$

of infinite cyclic subgroups (see [5], [17]). Then by Theorem 7.9 of [4], we conclude that for sufficiently large odd p, the quotient group

$$\Gamma' = \Gamma/\langle\gamma^p, (\gamma^n s_1)^p, \dots, (\gamma^{mn} s_m)^p\rangle$$

is non-elementary word hyperbolic. But Γ' is generated by elements of odd order; hence it has no subgroup of index two. □

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