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Chordal and Timbral Morphologies using Hamiltonian Cycles

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In this paper, we investigate several musical morphologies that can be represented as paths in abstract graphs. Our examples come from questions posed by composers in the compositional process. In particular, we focus on Hamiltonian paths and cycles, which are central to graph theory. Our results show the circumstances in which such a path exists in the graphs derived from these musical ideas.

Keywords: graph theory; musical form; musical shape; musical structure; morphology; Hamiltonicity; Tom Johnson

1. INTRODUCTION

Composers often implement enumerative processes to create musical morphologies (that is, an ordered set of musical elements). Apt examples of such techniques can be found in the work of Tom Johnson; notably, *The Chord Catalogue* (Johnson 1986), which enumerates through all 8178 possible chords of one octave. *The Chord Catalogue* is an example of a chordal morphology that is non-repeating and exhaustive. That is, every possible chord is played once and only once. What is perhaps the most interesting (and often least discussed/most overlooked) aspect of the piece is how the chords are ordered. First, all two-note chords are presented, then three, then four, etc. Within each of these sets, the chords are sorted in colexicographical order (where $(a, b) \leq (a', b')$ if and only if $b < b'$ or $b = b'$ and $a \leq a'$). For example, the two-note chords are ordered as follows:

Figure 1. The two-note chords in Johnson's *The Chord Catalogue*

In this paper, we discuss several non-repeating, exhaustive chordal and timbral morphologies with a focus on the ordering of the musical elements. In particular, we examine morphologies in which the number of elements that stay the same from event to event is specified. Each of our examples can be represented by a Hamiltonian path or cycle

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through a graph. Hamiltonian paths and cycles pass through each vertex once and only once, but a Hamiltonian cycle also ends at the start vertex. The vertices of the graph represent all chordal or timbral possibilities given certain combinatorial constraints and the edges represent possible movements from one event to the next that satisfy given morphological constraints.

This paper aims to provide interesting material for both musicians and mathematicians. We have structured it so that the mathematical and musical parts can be read independently. After long consideration, we decided that it was most effective to present both the musical and mathematical ideas in such a way that they are not tempered for any given audience. We hope that by doing so, we better illustrate the fact that musical questions can result in rich mathematical results and vice-versa.

In the next section, we start by discussing the musical and mathematical motivations of using graph theory to represent musical shapes and structures. Section 3 juxtaposes the musical and mathematical definitions of each of our examples. As mentioned above, either the musical or mathematical definitions can be read alone but we hope that presenting them side-by-side will provide insight into the intrinsic connections between the two domains. Section 4 gives further mathematical preliminaries needed for the proofs of the main theorems. The main results of the paper are Theorems 5.5, 5.14, 5.16, and 5.18, where we prove that certain graphs derived from musical problems are Hamiltonian or admit a Hamiltonian path. These theorems are introduced and proved in Section 5. This section, especially the proofs, are intended primarily for a mathematical audience. We conclude with a brief discussion regarding open musical and mathematical questions that were engendered by this research.

2. MATHEMATICAL AND MUSICAL MOTIVATIONS

In *Meta+Hodos* (Tenney 1986), James Tenney suggests that musical form consists of two primary components: *shape* and *structure*. The former being the relationships between adjacent elements (in time) and the latter the relationships of component parts to other parts and the whole (irrespective of time). While these are cogent definitions, they are not well-defined mathematically. Tenney, in collaboration with Larry Polansky, later developed a simple computer model of temporal gestalt perception using principles set forth in *Meta+Hodos*. In their article, “Hierarchical temporal gestalt perception in music: a metric space model” (Tenney and Polansky 1980), the authors explain the model and how they represent shape and structure in lay terms, but do not give exact mathematical formalization. Later, in “Morphological metrics” (Polansky 1996), which examines several ways of comparing musical shapes/morphologies, Polansky gives a strict mathematical definition of a morphology (along with musical applications).

> “A morphology (morph) is an ordered set M. The elements of M are identified as M_i, where i goes from 1 to L. L is the length of M... Morphs are ordered shapes such as melodies, duration series, harmonic orderings, spectra or statistical measures of formal segregation like the mean pitches of sections of a piece.”

As will be shown in more detail later, graph theory provides a convenient means of representing musical structure (the other factor of musical form according to Tenney). In short, the vertices of a graph represent musical elements and the edges represent relations between the elements. Edges can be induced by *a priori* definitions designed by the composer (as in the examples that follow) or as *a posteriori* observations based on the perception/experience/analysis of a piece. We call a definition that induces a connection between structural elements a *morphological constraint*.

Not only can a graph represent a musical structure, but graph theory also provides a formal link between shape and structure because a musical morphology can be represented as a path in a graph.

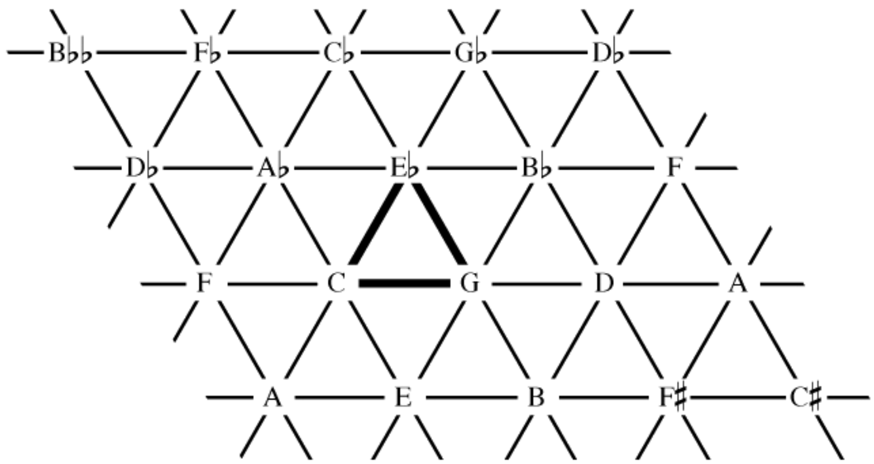

Figure 2. The neo-Riemannian *Tonnetz*

Perhaps the most canonical example of the connection between graph theory and music is the *Tonnetz* (or tone-network). Fig. 2 shows the neo-Riemannian *Tonnetz* where pitches are connected if they are related by either a major third, a minor third, or a perfect fifth. Movement by edge-adjacent triads satisfy parsimonious voice leading as two pitches stay the same while the other moves by a tone or a semitone. The *Tonnetz* conveniently provides a graph-based model of harmonic motion between chords as subject to the fundamental operations of neo-Riemannian music theory: P (substituting a triad by its parallel), R (substituting a triad by its relative), and L (substituting a triad for its leading tone). Fig. 3 shows an example of what Richard Cohn calls a "maximally smooth cycle" (Cohn 1996). One could easily chart this progression on the neo-Riemannian *Tonnetz* and, as will be shown later, many of our examples can be related to Cohn's notion of a maximally smooth cycle.

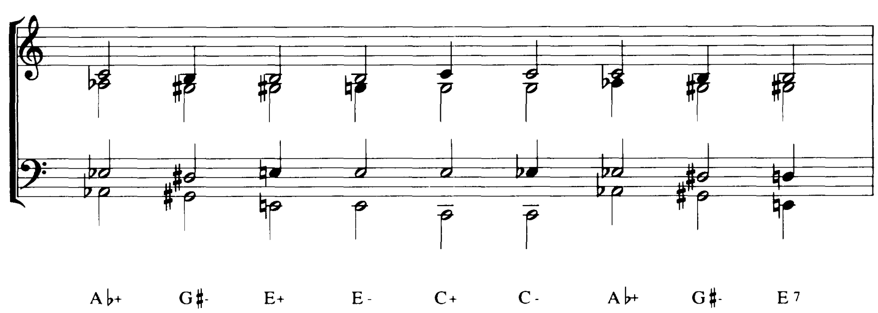

Figure 3. Richard Cohn's reduction of bars 268–79 from the first movement of Brahm' Concerto for Violin and Cello

Another, more recent, example is the harmonic lattice as employed by composers James Tenney and Ben Johnston. In a harmonic lattice (see Fig. 4), vertices are frequency

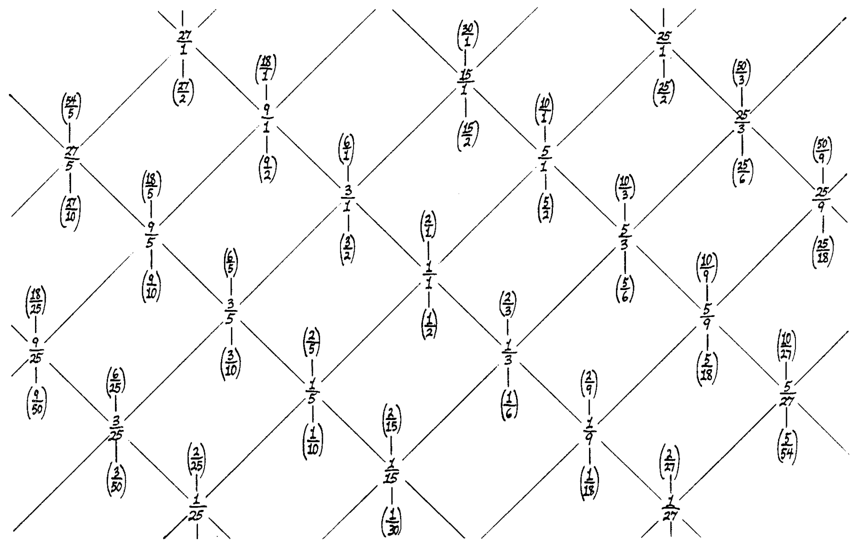

Figure 4. From James Tenney's *John Cage and the theory of harmony* (Tenney 1984): a harmonic lattice in the 3,5 plane of harmonic space

ratios with respect to a reference fundamental (1/1). The number of dimensions/axes correspond to the number of prime factors required to specify the frequency ratios and two frequency ratios are connected if they differ by one prime factor in either the numerator or denominator. Tenney suggests that the degree of separation between any two points on the lattice models our perception of distance in harmonic space. He has also written several pieces such as *Changes* for 6 harps (1985), *Water on the Mountain... Fire in Heaven* for 6 guitars (1985), and *Arbor Vitae* for string quartet (2006), where harmonic movement is determined by a random walk in harmonic space (for more details on these pieces, see Tenney 1987; Fiore 2013; Winter 2007, respectively).

Composers have long used mathematics to understand, analyze and generate music. Still, despite the strong precedence of the use of the *Tonnetz* to map triadic motion and the harmonic lattice to represent distance in harmonic space, the understanding of the connections between graph theory and music—especially beyond the study of more traditional tonal harmony—is still in progress (for more examples, see Crans, Fiore, and Satyendra 2008; Douthett and Steinbach 1998; Gollin 2000; Hook 2007; Hyer 1995; Lewin 1990, 2007; Mazzola 2003; Morris 2010; Tymoczko 2011; Waller 1978; Yust 2006).

In this paper, even though we hope to clearly explicate certain connections between music and graph theory and even though our examples can be related to certain aspects of neo-Riemannian analysis, we intentionally do not discuss the relation of elements to one another (as opposed to the functionality of musical elements defined in the theories of tonal harmony). In a way, we are presenting these ideas more in the abstract. In particular, we focus on graphs where the edges are induced by the number of common elements between vertices and Hamiltonian cycles on these graphs, which, as mentioned above, have the statistical property that every vertex occurs once and only once. The proofs of Hamiltonicity (the existence of a Hamiltonian cycle) on the defined graphs show that finding a given musical morphology is possible with little regard for the phenomenological implications of these compositional techniques or even how to find the paths (which is necessary in the compositional process). With that said, it is our contention that the musical shapes and structures examined in the following examples exhibit perceptual cohesion by virtue of a kind of conceptual clarity that is recapitulated by their rather simple, albeit very deep, mathematical formalizations.

The examples in this paper also elucidate two important facts: that graph theory is particularly applicable to the study of musical form and that there exists several musically motivated questions that can be of interest to mathematicians working in the field of graph theory. Further, we feel that the most effective way of better understanding the intersection between the two domains is via collaborations between mathematicians and musicians. This is exemplified through our personal collaborative experience and the fact that Hamiltonicity (the main focus of this paper) is central to the study of graphs even though the questions in this paper were originally musically motivated for compositional purposes.

The genesis of this research started with a musician's compositional problem (and a corresponding inability to express the problem mathematically) and a mathematician's mindset that interesting problems can be found in unlikely places. As we worked together, we developed a mode of communication where we could address our individual interests and problems. The result was a series of new musical works, several mathematical proofs, and a clear formalization of a particular way of connecting graph theory and musical form. The rather innocuous cry for help from the composer developed into much more. It warrants mention that while this publication post-dates the second author's dissertation entitled, "Structural Metrics: an Epistemology" (Winter 2010), our collaboration on these problems was actually the foundation for the dissertation (which goes beyond graph theory into yet other fields of mathematics such as algorithmic information theory). The dissertation actually cites the preprint of this paper. Point being is that we feel the connections between mathematics and music should be continually examined and re-examined with the hope to uncover deep problems in both fields.

3. MUSICAL EXAMPLES AND ASSOCIATED GRAPHS

Below, we define and motivate our main objects of study musically and mathematically, graphs $T_{n,k,l}$, $C_{n,k,l}$, J_n, and P_n. The graphs are presented in the order that we worked on them, however, this order also serves a pedagogical trajectory. The $C_{n,k,l}$-graphs were derived in consideration of both $T_{n,k,l}$ and another graph $J_{n,k,l}$ defined by Tom Johnson. All of the examples, including J_n and P_n, can be related to the traditional concept of the *Tonnetz* (as described above). The graphs J_n and P_n are motivated by an idea similar to Cohn's notion of maximally smooth cycles. However, between triads, one pitch stays the same and the other two move by contrary motion by a semitone. It is worth noting that we in no way suggest that these necessarily have application in neo-Riemannian analysis. Our musical examples are all from literature written in the last 10 years. The pieces (and thus the vertices in each graph) fully exhaust a combinatorial constraint that is not subject to any filtering based on functional harmony. For example, the vertices of J_n and P_n can be any triad and not necessarily those whose structure is based on major and minor thirds or any other intervallic constraint. Finally, throughout, we use the word 'pitch' to denote any pitch and not necessarily a pitch-class. We do not assume octave equivalence, however most of the graphs can be limited to a 12-tone set within one octave. Only J_n and P_n specifically refer to a set of pitches. As for the others, the numbers or indices of the tuple can be bijectively mapped onto any set of distinct pitches.

3.1. Timbral graph $T_{n,k,l}$ of voicings for a chord with k pitches using n instruments where l pitches are assigned the same instrument in both voicings of any adjacent pair

We define a timbral morphology that enumerates without repetition through all timbral combinations of an arbitrary static chord given a morphological constraint that defines the number of pitches played by the same instrument from chord to chord. That is, the same chord is repeated but the mapping of instruments to pitches changes. Such a morphology is analogous to a Hamiltonian path in the following graph. Each vertex represents a k-tuple populated with symbols from an alphabet of a given size (repeats in the tuple are allowed because we assume that each instrument can perform any number of pitches at once). Edges are induced between vertices that satisfy a morphological constraint, defined as the number of positions populated by the same symbol from tuple to tuple. For our purposes, each position in the tuple represents a pitch in the arbitrary static chord to which an instrument (represented by a unique symbol from the alphabet) is assigned. We denote this graph as $T_{n,k,l}$ where n is the number of distinct timbres/instruments (the alphabet size), k is the number of pitches in the given chord (the tuple size), and l is the number of pitches that are played by the same instrument from chord to chord. For example, the piece, *maximum change* (Winter 2007), is an example of $T_{4,4,0}$. In *maximum change*, a vertex represented by $(1, 2, 3, 4)$ may denote that C, D, E, F♯ be performed by crotales (1), glockenspiel (2), chimes (3), and piano (4), respectively. The morphological constraint is that each pitch is assigned a different instrument upon each successive event (see Fig. 5 for the first 10 chords of *maximum change*). For example, the vertex representing $(1, 2, 3, 4)$ is connected via edges to $(2, 1, 1, 1), (2, 3, 2, 2)$, etc. This graph has 256 vertices representing all timbral combinations of a static chord with 4 different pitches played by 4 instruments of distinct timbre. The 10368 edges are induced by the morphological constraint of maximum timbral change (total derangement) from event to event. Since this graph can be shown to be Hamiltonian, a non-repeating, exhaustive enumeration of these timbral combinations given the morphological constraint is possible. We show that $T_{n,k,l}$ is Hamiltonian for all $n \geq 3$ and $k \geq l + 1$ (See Theorem 5.5).

Figure 5. Score excerpt of the first 10 chords of *maximum change* which is generated from $T_{4,4,0}$ (note that from measure to measure each pitch is passed to a different instrument)

Mathematical Definition of $T_{n,k,l}$. Let n, k, l be non-negative integers satisfying the condition $k \geq \max\{1, l\}$. The set of vertices of $T_{n,k,l}$ will consist of the set of all k-tuples $(x_1, x_2, \ldots, x_k)$ where $x_i \in \{1, 2, \ldots, n\}$. Two such k-tuples $\alpha = (x_1, x_2, \ldots, x_k)$ and $\beta = (y_1, y_2, \ldots, y_k)$ are connected by an edge if and only if $x_i = y_i$ for exactly l

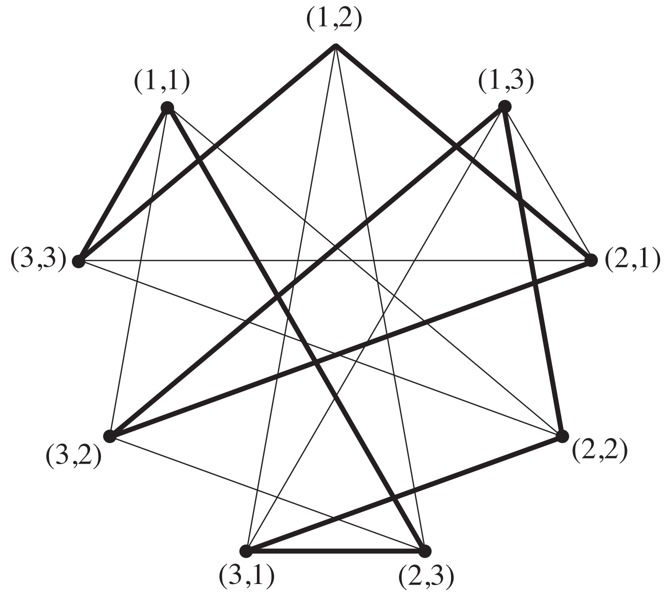

Figure 6. Graph of $T_{3,2}$ with a Hamiltonian cycle

values of $i \in \{1, 2, \ldots, k\}$. We also will denote the graph $T_{n,k,0}$ by $T_{n,k}$ (an example of $T_{3,2}$ is given in Fig. 6). In Theorem 5.5, we will prove that the timbral graph $T_{n,k,l}$ is Hamiltonian for all $n \geq 3$ and $k \geq l + 1$.

3.2. Chordal $C_{n,k,l}$-graphs with chords of size k from n possible pitches where l pitches are assigned the same instrument in both chords of any adjacent pair

Before introducing $C_{n,k,l}$-graphs, we explain a related chordal morphology proposed by Tom Johnson in "Musical questions for mathematicians" (Johnson 2005a). Johnson's morphology enumerates through chords of equal size in which the morphological constraint is the number of pitches that change, or conversely, repeat, from chord to chord. He gives an example stating, "A five-note scale contains $\binom{5}{3} = 10$ three-note chords. To have maximum harmonic movement, one might wish to require that there be two new notes as one passes from one chord to another." Such a morphology is analogous to a Hamiltonian path in a graph we denote as $J_{n,k,l}$. In $J_{n,k,l}$, each vertex is a k-element subset of some specified set of n possible pitches (alphabet size), k is the number of pitches in the chords (set size), and l is the number of pitches that repeat from chord to chord. As opposed to $T_{n,k,l}$, in $J_{n,k,l}$, symbols from the alphabet represent pitches rather than instruments and duplicates in a k-set are not allowed, for otherwise the chord would have duplicate pitches. Johnson's aforementioned example, $J_{5,3,1}$, is isomorphic to the Petersen graph and contains a Hamiltonian path but not a cycle (see Fig. 7 and 8). Coincidentally, the instance of $J_{n,k,l}$ where $l = k - 1$ is well known in mathematics as a Johnson Graph (after the mathematician Selmer M. Johnson). Of particular interest is the case of $l = 0$, called a Kneser Graph, the Hamiltonicity of which is an open, difficult problem.

As an alternate method to approach the general Hamiltonicity problem of $J_{n,k,l}$, we explore the Hamiltonicity of a variant graph that represents a chordal morphology with the following morphological constraint. Any number of pitches less than the chord size may repeat from chord to chord, but a certain number of them must be played by the same instrument. In this setting, each instrument plays exactly one pitch of the chord and

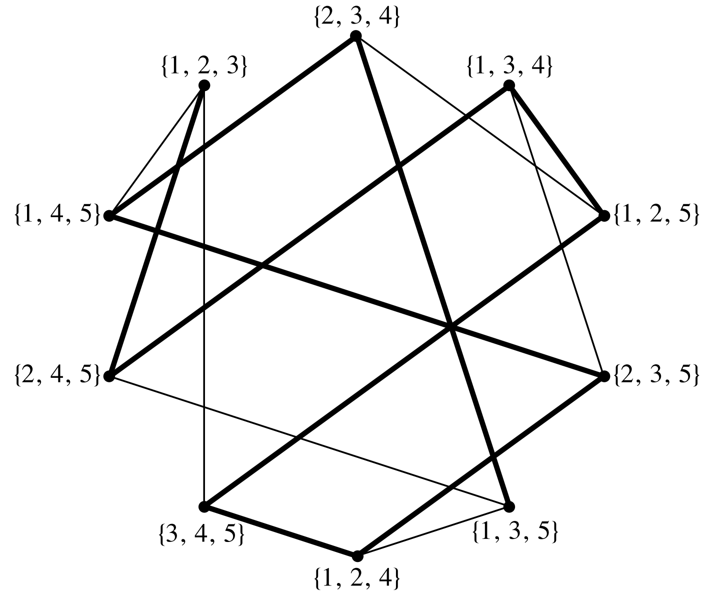

Figure 7. $J_{5,3,1}$ and a Hamiltonian path (in bold)

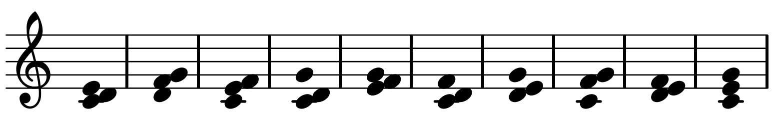

Figure 8. Chordal morphology of $J_{5,3,1}$ where 1, 2, 3, 4 and 5 are mapped to c, d, e, f and g, respectively

no pitch is simultaneously played by two instruments. We call such a graph a $C_{n,k,l}$-graph where n is the number of possible pitches, k is the chord size, and l is the number of pitches taken by the same instrument from chord to chord. Positions within the tuple indicate a timbre or instrument to which a pitch, indicated by a unique symbol from the symbol set, is assigned. This is the exact opposite of $T_{n,k,l}$, and as with $J_{n,k,l}$, duplicates within a tuple are not allowed. Further, since the enumeration is through all chordal possibilities with chords of size k, permutations of any given tuple are considered representatives of which only one will occur in the graph. For $C_{n,k,l}$-graphs, while $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, and $(3, 2, 1)$ each represents a different timbral arrangement of the same chord, only one of them will occur because the same group of pitches will never occur twice in the morphology. For example, if $(1, 2, 3)$ occurs, all other permutations of that tuple will not be used. Note that the choice of representatives greatly alters the edges of the graph.

A $C_{n,k,l}$-graph can be viewed in relation to $J_{n,k,l}$. In $J_{n,k,l}$, order does not matter so derangements are not allowed, but in a $C_{n,k,l}$-graph, order matters and derangements are allowed. For example, the vertices representing $(1, 2, 3, 4)$ and $(1, 3, 2, 5)$ are connected in a $C_{5,4,1}$-graph. As order does not matter for $J_{n,k,l}$, the corresponding unordered sets of $\{1, 2, 3, 4\}$ and $\{1, 2, 3, 5\}$ are connected in $J_{5,4,3}$ but not in $J_{5,4,1}$. So in a $C_{5,4,1}$-graph, even though one element is fixed, the two vertices actually have $k-1$ elements in common as in $J_{5,4,3}$. If one were to limit the number of elements in common from chord to chord or vertex to vertex in a $C_{n,k,l}$-graph, the graph would become harder to prove Hamiltonian. Without putting limits on the number of elements in common from vertex to vertex (they can be anywhere from l to $k-1$), we show in Theorem 5.14 that for some choice of representatives, a certain $C_{n,k,l}$-graph is Hamiltonian for all $n \geq 4, n \geq k+1, k \geq 2l$.

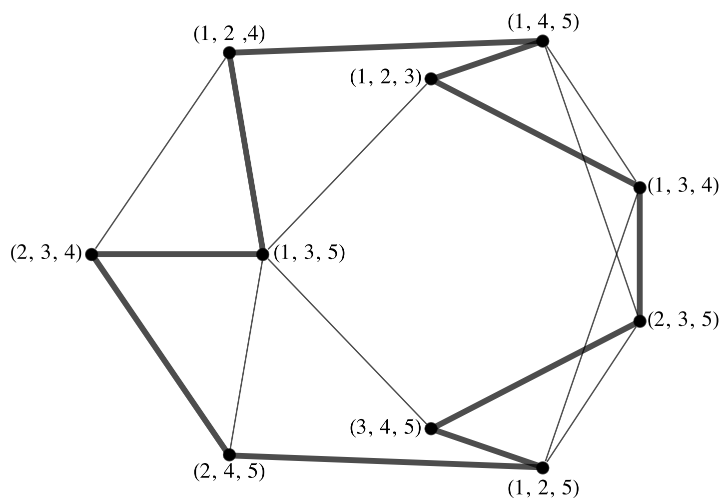

Figure 9. Graph and Hamiltonian path of a $C_{5,3,1}$-graph

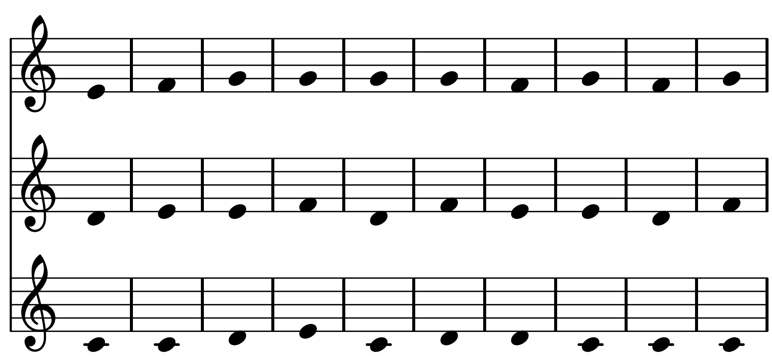

Figure 10. Chordal morphology of a $C_{5,3,1}$-graph where 1, 2, 3, 4 and 5 are mapped to c, d, e, f and g, respectively

Mathematical Definition of a $C_{n,k,l}$-graph. Similarly to the musical definition above, we first give the definition of $J_{n,k,l}$. Let $0 \leq l \leq k \leq n$. The set of vertices of $J_{n,k,l}$ consists of all k-subsets of the set $\{1, \dots, n\}$. Two k-subsets are connected by an edge if and only if their intersection has size l, i.e. it is an l-subset of $\{1, \dots, n\}$.

A $C_{n,k,l}$-*graph* is defined as follows. Let $0 \leq l \leq k \leq n$ where $k \geq 1$. Let also $A_n = \{1, 2, \dots, n\}$, and let $A_{n,k}$ be the set of all ordered k-tuples $(x_1, x_2, \dots, x_k)$, $x_i \in A_n, x_i \neq x_j$ for all $i \neq j$. *A choice of representatives* π is a subset of $A_{n,k}$ such that for each k-subset S of A_n, π contains exactly one permutation of S. This permutation will be denoted by $\pi(S)$.

The set of vertices of a $C_{n,k,l}$-graph is in bijective correspondence with the set of all k-subsets of A_n. The set of edges will depend on the choice of representatives. Let π be a choice of representatives, and let α, β be two distinct k-subsets,

$\pi(\alpha) = (x_1, x_2, \ldots, x_k), \pi(\beta) = (y_1, y_2, \ldots, y_k)$. Then, in the graph $C_{n,k,l}(\pi)$, the vertices α and β are connected by an edge if and only if $x_i = y_i$ for exactly l values of $i \in \{1, 2, \ldots, k\}$. For brevity, we will denote the graph $C_{n,k,0}(\pi)$ by $C_{n,k}(\pi)$. Thus, in a $C_{n,k}$-graph, two k-tuples are connected by an edge if and only if the entries differ at every spot. If two ordered k-tuples from $A_{n,k}$ differ at every spot (entry), then we will say that each one is *a total derangement* of the other.

In Theorem 5.14, we will prove that for all non-negative integers n, k and l, with $n \geq \max\{4, k+1\}$ and $k \geq \max\{2l, 1\}$, there is a choice of representatives π such that $C_{n,k,l}(\pi)$ is Hamiltonian.

Examples of $C_{n,k,l}$-graphs.

1. Let $n = 3, k = 2, l = 1$, and let π be the following choice of representatives: $\pi(\{1,2\}) = (1,2), \pi(\{2,3\}) = (2,3), \pi(\{1,3\}) = (3,1)$. Then the graph $C_{3,2,1}(\pi)$ is easily seen to be isomorphic to the graph with 3 vertices and no edges.

2. Let π_0 denote the choice of representatives defined as follows: for every k-subset $\{x_1, x_2, \ldots, x_k\}, \pi_0(\{x_1, x_2, \ldots, x_k\}) = (y_1, y_2, \ldots, y_k)$ where $y_1 < y_2 < \ldots < y_k$. π_0 will be called *the natural choice of representatives.* It is easy to see that $C_{3,2,1}(\pi_0)$ is isomorphic to the tree with 3 vertices (such a tree is unique up to isomorphism). The graph $C_{4,2,1}(\pi_0)$ is a connected graph with 6 vertices and 8 edges, in particular, it is not a tree.

3.3. Chordal graphs J_n and P_n of chords with 3 pitches from a set of n possible pitches where one pitch stays the same and the others move by a semitone in contrary motion between adjacent chords

In Tom Johnson's piece, *Trio* (Johnson 2005b), the musical morphology is derived from a Hamiltonian path in a graph where vertices are analogous to chords with three pitches represented by numbers from 0 to 48, where middle C equals 24. The numbers of each chord are distinct partitions (without repetitions) of 72. That is, the constituent numerical representation of the pitches of each chord sum to 72. The edges are induced by the morphological constraint that from chord to chord, one pitch must remain the same while the other pitches move by a semitone in contrary motion. Fig. 11 shows the final morphology with the numeric representation of the pitches in each chord. Note that this graph does not show all the edges induced by the morphological constraint. Fig. 12 shows the first system of the score.

The final graph we investigate is a generalization of the above graph, which we define as J_n where n is the size of the integer to be partitioned (J_n has no relation to $J_{n,k,l}$). The number of partitions (the tuple size), as with Johnson's definition, is always 3 and the morphological constraint remains that from vertex to vertex, one integer remains the same and the other two change by 1 in opposite directions. It is important to remark that, musically, the order of the elements in the tuple does not matter as they represent pitches (for example, $(23, 24, 25)$ is musically equivalent to $(24, 25, 23)$); however, Johnson created the graph with tuples ordered from low to high and our mathematical definition of J_n reflects and requires that the orderings are preserved.

We show that (Theorem 5.16) for all $n \geq 6$, the graph J_n contains a Hamiltonian path but not a cycle. We recognize that this is a weaker result than finding a Hamiltonian cycle as we do with all the other graphs we have defined thus far. Therefore we also define the graph P_n where the tuple can be in any order. The definition of P_n is equally true to Johnson's musical idea, but we introduced J_n first to explain why Johnson found a Hamiltonian path rather than a Hamiltonian cycle. In Theorem 5.18, we show P_n contains a Hamiltonian cycle for all $n \geq 8$.

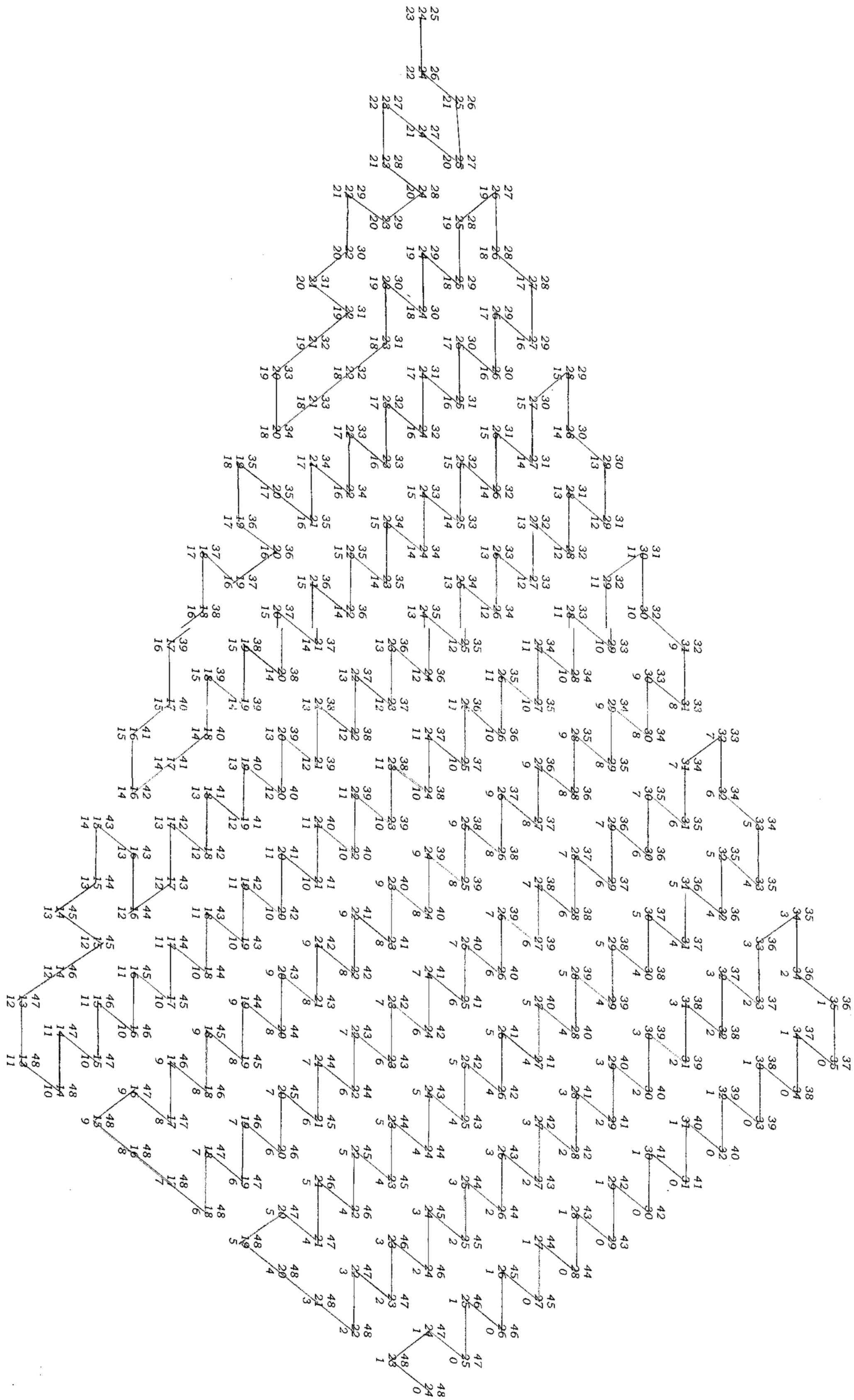

Figure 11. *Trio* graph (from cover of score)

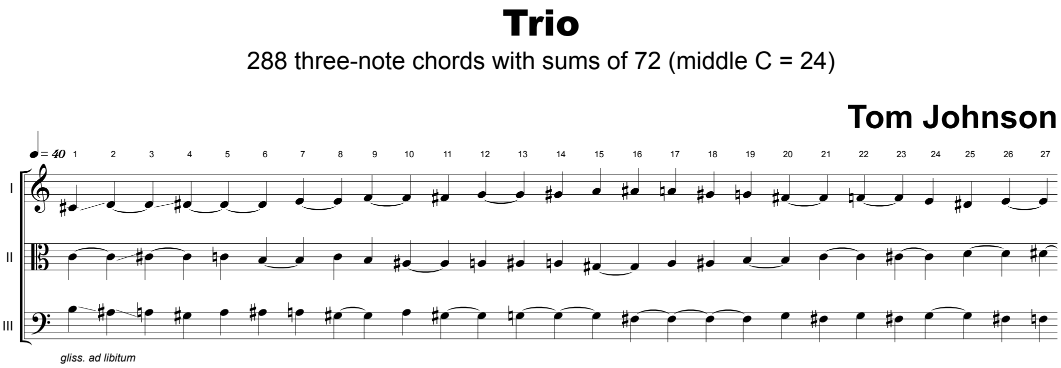

Figure 12. First system of *Trio* graph

Mathematical Definition of J_n. For any $n \geq 6$, the graph J_n has the vertex set $V_n = \{(x, y, z) \mid x, y, z \in \mathbb{N},\ x < y < z,\ x + y + z = n\}$ where $\mathbb{N}$ denotes the set of positive integers. Two vertices $(x_1, x_2, x_3), (y_1, y_2, y_3) \in V_n$ are connected if and only if $x_i = y_i$ for some $i \in \{1, 2, 3\}$ and $x_j = y_j + 1$ for some $j \in \{1, 2, 3\} \backslash \{i\}$. Note that then $x_k = y_k - 1$ for $k \in \{1, 2, 3\} \backslash \{i, j\}$.

Mathematical Definition of P_n. For any $n \geq 6$, the graph P_n has the vertex set $V_n = \{(x, y, z) \mid x, y, z \text{ are distinct positive integers, and, } x + y + z = n\}$. The edges are defined as in the case of J_n, that is two vertices $(x_1, x_2, x_3), (y_1, y_2, y_3) \in V_n$ are adjacent if and only if $x_i = y_i$ for some $i \in \{1, 2, 3\}$ and $x_j = y_j + 1$ for some $j \in \{1, 2, 3\} \backslash \{i\}$.

The set of vertices of P_n can be identified with the "triangular subset" τ_n of the integer lattice $\mathbb{Z}^2$ defined as follows:

$$\tau_n = \{(x, y) \mid x, y \in \mathbb{N},\ x, y, n - (x + y) \text{ are distinct and } x + y \leq n - 1\}$$

The set of edges is defined as follows: a vertex $(x, y) \in \tau_n$ is connected to the vertices $(x, y + 1), (x, y - 1), (x - 1, y), (x + 1, y), (x + 1, y - 1), (x - 1, y + 1)$ provided that, in each case, the latter is in τ_n.

We will denote $\Delta_n = \{(x, y) \mid x, y \in \mathbb{N},\ x + y \leq n - 1\}$.

The vertices $(1, 2)$ and $(1, 3)$ are connected by an edge in the graph P_n for all $n \geq 8$; the edge $((1, 2), (1, 3))$ will be called *the special edge* of P_n.

4. MATHEMATICAL PRELIMINARIES NEEDED FOR THE MAIN THEOREMS

For very basic notions of graph theory we refer the reader to (Cameron 1995). All the graphs in this paper will be finite and simple (in a simple graph, loops and multiple edges are not allowed).

Definition 4.1 If $G = (V, E)$ is a graph, then any subset $A \subset V$ defines a subgraph denoted by $G(A) = (A, E(A))$ where $E(A) = \{(u, v) \mid u, v \in A, (u, v) \in E\}$. The graph $G(A)$ is called *the full subgraph of* G with respect to A (or a full subgraph of G on A).

Given any positive integer n, *a complete graph* on n vertices denoted by K_n is defined as a simple graph where every pair of distinct vertices is connected by an edge, and *a cyclic graph* on n vertices denoted by C_n is defined as a connected graph where the degree of every vertex is two (such a graph is unique up to isomorphism).

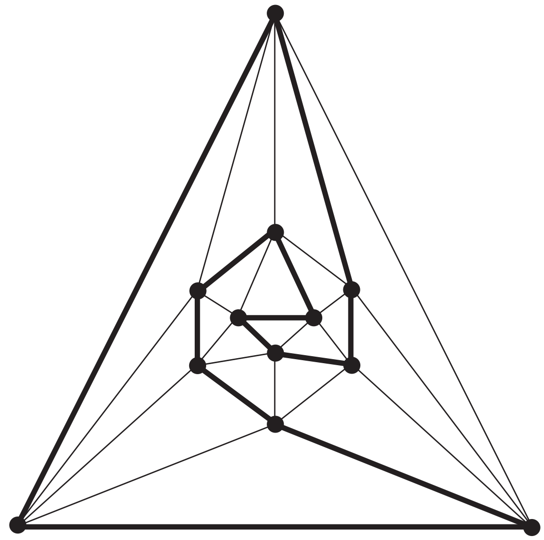

Figure 13. Graph of an icosahedron with a Hamiltonian cycle

Definition 4.2 A graph $G = (V, E)$ is *Hamiltonian* if it has a cycle of length equal to $|V|$; such a cycle is called a *Hamiltonian cycle*. Similarly, a graph is *traceable* if it has a path of length $|V|$; such a path is called a *Hamiltonian path*.

Both K_n and C_n are obvious examples of Hamiltonian graphs. Let us emphasize that having a Hamiltonian path is weaker than being Hamiltonian, i.e. being Hamiltonian requires having a path visiting each vertex exactly once but also returning back to the original vertex. There are many examples of even regular graphs (such as the Petersen Graph, see Cameron 1995, and Fig. 7) which contain a Hamiltonian path but are not Hamiltonian. The study of Hamiltonian graphs is one of the central problems in graph theory. No complete description of such graphs is available. There are some interesting sufficiency criteria for Hamiltonicity, the most classical one being Dirac's Theorem, which states that if $G = (V, E)$ is a graph where the degree of every vertex is at least $|V|/2$ then G is Hamiltonian (again, see Cameron 1995). But very often all the known criteria are too weak to apply in concrete examples. For example, one often deals with graphs where the degrees of vertices are uniformly bounded by some number r that is extremely small by comparison to the number n of vertices in the graph.

Definition 4.3 Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ be two graphs. The *tensor product* $G = (V, E)$ of these graphs is defined as follows: $V = V_1 \times V_2, E = \{((x, y), (z, t)) \mid (x, z) \in E_1, (y, t) \in E_2\}$.

Example 4.4 [Tensor Product] The tensor product of T_{n,k_1} and T_{n,k_2} is isomorphic to T_{n,k_1+k_2}.

Definition 4.5 Let H denote the set of all Hamiltonian graphs $G = (V, E)$ such that if $|V|$ is even then G contains a Hamiltonian cycle $C = (v_1, v_2, \ldots, v_{2n})$ such that there exists $i, j, k, l \in \{1, 2, \ldots, 2n\}$ with i, j even, k, l odd and $(v_i, v_j), (v_k, v_l) \in E$. In particular, if $|V|$ is odd and G is Hamiltonian, then G belongs to H.

Example 4.6 (Elements of $\boldsymbol{H}$) **1.** The graph C_n - the n-cycle - is Hamiltonian for all $n \in \mathbb{N}$ but belongs to the class H if and only if n is odd.
2. The complete graph K_n belongs to the class H if and only if $n \geq 3$.

The following proposition, proved in (Gravier 1997), will be used heavily in the proof of Theorem 5.5.

Proposition 4.7 *The tensor product of two Hamiltonian graphs G_1 and G_2 is Hamiltonian if and only if at least one of the factors belongs to H.*

Definition 4.8 Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ be two graphs. The *Cartesian product* $G = (V, E)$ of these graphs is defined as follows,

$$V = V_1 \times V_2,\ E = \{((x, y), (z, t)) \mid \text{either } x = z, (y, t) \in E_2 \text{ or } (x, z) \in E_1, y = t\}$$

Example 4.9 (Cartesian Product.) The graph $T_{n,2,1}$ is isomorphic to the Cartesian product of two complete graphs on n vertices.

It is a well known fact (and easy to prove) that the Cartesian product of Hamiltonian graphs is Hamiltonian.

We also will need the following classical result in combinatorics.

Lemma 4.10 **(Hall's Marriage Lemma)** *Let $A = (a_{ij}), 1 \leq i, j \leq n$ be an $n \times n$ matrix with all entries either 0 or 1. Let also $r \in \{1, \ldots, n\}$, $1 \leq x_1 < x_2 < \ldots < x_r \leq n$ where $x_i \in \mathbb{Z}$. Assume that for every such choice of r and $x_1, \ldots, x_r$ the cardinality of the set $\{p \mid 1 \leq p \leq n,\ \exists j \in \{1, \ldots, r\}$ such that $a_{px_j} = 1\}$ is at least r. Then one can choose n entries of the matrix A such that all of the chosen entries are 1 and no two of them are in the same column or in the same row.*

See (Cameron 1995) for a proof of Lemma 4.10.

Hall's Marriage Lemma can be reformulated as follows. Assume there are n men and n women such that for each $1 \leq r \leq n$, each set of r women like at least r men. Then one can match the women to the men so that we have exactly n couples (marriages).

5. MAIN THEOREMS

In this section we will prove Hamiltonicity results for the graphs $T_{n,k,l}$, $C_{n,k,l}$, J_n, and P_n.

5.1. *The Hamiltonicity of $T_{n,k,l}$ for $n \geq 3$ and $k \geq l + 1$.*

Lemma 5.1 *The graph $T_{n,2}$, $n \geq 3$ belongs to H.*

Proof. The degree of every vertex of $T_{n,2}$ equals $(n-1)^2$ while the order of the graph equals n^2. Since $(n-1)^2 \geq \frac{1}{2}n^2$ for all $n \geq 4$, by Dirac's Theorem we conclude that $T_{n,2}, n \geq 4$ is Hamiltonian. On the other hand, the graph $T_{3,2}$ is also Hamiltonian (see Fig. 6). Then, if n is odd then $T_{n,2}$ contains an odd number of vertices therefore it belongs to the class H.

Assume now $n = 2k, k \geq 2$. For $k = 2$ the graph $T_{4,2}$ is easily seen to belong to the class H. Let us assume $k \geq 3$. We can represent the vertices of $T_{2k,2}$ by the points $(i, j), 1 \leq i, j \leq n, i, j \in \mathbb{Z}$ in the plane where the points $(i, j), (i', j')$ are connected by an edge if and only if $i \neq j, i' \neq j'$. Let $C_1 = ((k+1, k+1), (k+2, k+2), \ldots, (n, n)), C_2 = ((1, 1), (2, 2), \ldots, (k, k))$, $UD_i = ((1, i), (2, i+1), \ldots, (n-i+1, n))$ be the i-th diagonal above the main diagonal, and $LD_i = ((i, 1), (i+1, 2), \ldots, (n, n-i+1))$ be the i-th diagonal below the main diagonal. Then we observe that $C_1, C_2, LD_2, LD_3, \ldots, LD_{n-1}, UD_n$, $LD_n, UD_2, UD_3, \ldots, UD_{n-1}$ is a Hamiltonian path in $T_{n,2}$. Moreover, the first vertex of C_1 (the point $(k+1, k+1)$) and the last vertex of UD_{n-1} (the point $(2, n)$)

are connected thus we have a Hamiltonian cycle. Furthermore, since the vertices $(1,1),(2,2),(3,3),\dots,(n,n)$ are all mutually connected, we conclude that $T_{n,2}$ belongs to H. ■

Lemma 5.2 *The graph $T_{n,2,1}, n \geq 3$ belongs to class H in 4.5*

Proof. The $T_{n,2,1}$ is isomorphic to the Cartesian product of two K_n's therefore it is Hamiltonian. On the other hand, $T_{n,2,1}$ contains n^2 vertices therefore if n is odd, $T_{n,2,1}$ belongs to H.

Assume now $n = 2k, k \geq 2$. We can represent the vertices of $T_{2k,2,1}$ by the points $(i,j), 1 \leq i,j \leq n, i,j \in \mathbb{Z}$ in the plane where the points $(i,j),(i',j')$ are connected by an edge if and only if either $i = i', j \neq j'$ or $i \neq i', j = j'$.

Let $D_i = ((i,1),(i,2),\dots,(i,n))$ denote "the path along the n-th row". Then it is easy to see that $D_1 D_2 \dots D_n$ is a Hamiltonian cycle. Let $D = (v_1,\dots,v_{n^2},v_1)$ denotes this cycle. Then $v_1 = (1,1), v_{2n} = (1,2), v_{3n} = (1,3), v_{4n} = (1,4)$. Note that any two of these four vertices are connected by an edge thus we obtain that $T_{n,2,1}$ belongs to H. ■

Proposition 5.3 *The graph $T_{n,k}$ is Hamiltonian for all $n \geq 3$.*

Proof. We proceed by induction on k. For $k = 1$, $T_{n,1}$ is Hamiltonian for $n \geq 3$ since it is isomorphic to K_n, the complete graph of order n; and $T_{n,2}, n \geq 3$, is Hamiltonian by Lemma 5.1. Let us assume that $k > 2$ and the graph $T_{n,m}$ is Hamiltonian for all $m < k$.

Let us first assume that n is odd. Note that $T_{n,k}$ is the tensor product $T_{n,k-1}$ and $T_{n,1}$. Moreover, $|V(T_{n,m})| = n^m$. The graph $T_{n,k-1}$ is Hamiltonian by inductive hypothesis. Also, since its order is odd, $T_{n,k-1}$ belongs to H. Then by Proposition 4.7, $T_{n,k}$ is Hamiltonian.

When n is even, by Example 4.4, $T_{n,k}$ is the tensor product of $T_{n,2}$ and $T_{n,k-2}$. By Lemma 5.1, $T_{n,2}$ belongs to H, and by inductive hypothesis, $T_{n,k-2}$ is Hamiltonian. Then, by Proposition 4.7, we conclude that $T_{n,k}$ is Hamiltonian. ■

Let us also observe that the graph $T_{n,l+1,l}, n \geq 3$ is isomorphic to a Cartesian product of K_n's; thus we obtain the following lemma.

Lemma 5.4 *The graph $T_{n,l+1,l}, n \geq 3$ is Hamiltonian for every non-negative integer l.*

Theorem 5.5 *The graph $T_{n,k,l}$ is Hamiltonian for all $n \geq 3$ and $k \geq l+1$. In particular, the graph $T_{n,k}$ is Hamiltonian for all $n \geq 3$.*

Proof. The proof is by induction on $k - l$.

If $k - l = 1$ then the claim follows from Lemma 5.4. Now let $k - l \geq 2$. By Proposition 5.3, we may also assume that $l \geq 1$.

Then we observe that the graph $T_{n,k,l}$ contains a subgraph isomorphic to the tensor product of the graphs $T_{n,k-2,l-1}$ and $T_{n,2,1}$ (because every k-tuple can be viewed as a pair of $(k-2)$-tuple and a 2-tuple). The graph $T_{n,k-2,l-1}$ is Hamiltonian by inductive assumption. The graph $T_{n,2,1}$ is Hamiltonian and belongs to the class H as shown by Lemma 5.2. Then, by Proposition 4.7, we conclude that $T_{n,k,l}$ is Hamiltonian. ■

Remark 5.6 Note that in the case $l = k$ and $n > 1$, the graph $T_{n,k,l}$ has no edges, so is disconnected and not Hamiltonian. Also, the graph $T_{1,k,l}, k \geq 1$ consists of a single vertex with no edge. The graph $T_{2,k,0}$, however, is not Hamiltonian for $k \geq 2$ since it contains $2^k \geq 4$ vertices and every vertex in this graph is connected to exactly one other vertex since each k-tuple consisting of 1's and 2's is connected to the k-tuple where "1" is replaced by "2" and "2" is replaced by "1". The case $T_{2,k,l}$ for general k and l, $k > l$, is more complex.

5.2. *There is a choice of representatives π such that $C_{n,k,l}(\pi)$ is Hamiltonian.*

For a $C_{n,k,l}$-graph, again, the case $l = 0$ is special. First, we need the following lemma.

LEMMA 5.7 *Let $n \geq \max\ \{4, k\}$, and $\alpha, \beta, \gamma \in A_{n,k}$ be ordered k-tuples. Then there exists an ordered k-tuple $\delta \in A_{n,k}$ such that δ is a permutation of γ, and δ is also a total derangement of α and of β.*

Proof. Let $\alpha = (a_1, \ldots, a_k), \beta = (b_1, \ldots, b_k), \gamma = (c_1, \ldots, c_k)$ and $\delta = (z_1, \ldots, z_k)$. The condition in the lemma is equivalent to the following: δ is a permutation of γ and $z_i \notin \{a_i, b_i\}$ for all $i \in \{1, \ldots, k\}$. So for each value z_i, only two values, namely, a_i and b_i, are not acceptable and all the other $n-2$ values can be chosen. But since $n - 2 \geq 2$, by Marriage Lemma, we can arrange it. ■

Remark 5.8 The condition $n \geq 4$ is sharp. Indeed, for $n = 3, k = 2$, if $\alpha = (1, 2), \beta = (1, 3), \gamma = (2, 3)$, then there is no δ satisfying the conditions of the lemma.

COROLLARY 5.9 *Let $n \geq \max\ \{4, k\}$, and $\alpha, \gamma \in A_{n,k}$ be ordered k-tuples. Then there exists a k-tuple $\delta \in A_{n,k}$ such that δ is a permutation of γ, and δ is also a total derangement of α.*

PROPOSITION 5.10 *Let n, k be positive integers with $n \geq \max\{3, k+1\}$.*
a) For some choice π of representatives, the graph $C_{n,k}(\pi)$ is Hamiltonian.
b) Let $n \geq 4, \alpha, \beta$ be two distinct k-subsets of A_n, and the ordered k-tuples $(a_1, \ldots, a_k), (b_1, \ldots, b_k)$ be permutations of α and β respectively. Then there exists a choice of representatives π and Hamiltonian path in $C_{n,k}(\pi)$ starting at α, ending at β, and such that $\pi(\alpha) = (a_1, \ldots, a_k), \pi(\beta) = (b_1, \ldots, b_k)$.

Proof. a) Let $n \geq 4$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$ be all k-subsets of A_n (so $N = \binom{n}{k}$). Let also $\pi(\alpha_1)$ be any permutation of α_1. By Corollary 5.9, we can choose a permutation $\pi(\alpha_2)$ of α_2 which is a total derangement of $\pi(\alpha_1)$ and by induction, at each step i, we can define a permutation $\pi(\alpha_i)$ of α_i to be a total derangement of $\pi(\alpha_{i-1})$. For the last step, by Lemma 5.7, we can define a permutation $\pi(\alpha_N)$ of the last remaining k-subset α_N such that it is a total derangement of $\pi(\alpha_{N-1})$ and $\pi(\alpha_1)$.

For the case $n < 4$, we need to consider the graphs $C_{3,1}, C_{3,2}$. The graph $C_{3,1}$ is cyclic and therefore Hamiltonian. For the graph $C_{3,2}$, it is easy to find a choice of representative π and a Hamiltonian cycle, e.g. $(1, 2) \to (2, 3) \to (3, 1) \to (1, 2)$.

b) The claim follows immediately from the proof of part a). ■

Now we need to treat the case $l \geq 1$. First, we will consider a special case.

PROPOSITION 5.11 *For all integers k, n with $k \in \{2, 3\}$ and $n \geq 4$, there exists a choice of representatives π such that the graph $C_{n,k,1}(\pi)$ is Hamiltonian.*

Proof. The proof is by induction on n. For $n = 4$, let us observe that $C_1 = ((1, 2, 3), (1, 3, 4), (2, 1, 4), (2, 4, 3), (1, 2, 3))$ is a Hamiltonian cycle in a $C_{4,3,1}$-graph, and $C_2 = ((1, 2), (1, 3), (2, 3), (2, 4), (3, 4), (1, 4), (1, 2))$ is a Hamiltonian cycle in a $C_{4,2,1}$-graph.

For the step of the induction, let us assume that the claim holds for some $n \geq 4$. We will treat the cases $k = 2$ and $k = 3$ separately.

Case 1: $k = 2$.

Let $C = ((1, 2), \ldots, (a, b), (1, 2))$ be a Hamiltonian cycle in a $C_{n,2,1}$-graph for some choice of representatives (without loss of generality, we may assume that the first vertex in a cycle is an ordered pair $(1,2)$). Then either $a \neq 1$ or $b \neq 2$. Assume that $a \neq 1$ (the case $b \neq 2$ is similar). Then we obtain a Hamiltonian cycle $((1,2),\ldots,(a,b),(a,n+1),\ldots(1,n+1))$ where the path $((a,n+1),\ldots(1,n+1))$ visits every vertex of the form $(i,n+1), 1 \leq i \leq n$.

Case 2: $k = 3$.

Again, let $C = ((a,b,c),\ldots,(1,2,n))$ be a Hamiltonian path in a $C_{n,3,1}$-graph for some choice of representatives. Let $d \in A_n \backslash \{b\}$, and S be the set of all 3-subsets of A_{n+1} which contain $n+1$, i.e. $S = A_{n+1,3} \backslash A_{n,3}$. We can make a choice of representatives on S to obtain a choice of representatives π on the whole set $A_{n+1,3}$ such that in the full subgraph of $C_{n+1,3,1}(\pi)$ on the set S, there exists a Hamiltonian path C' starting at an ordered triple $(1,3,n+1)$, and ending at an ordered triple $(a,d,n+1)$. Then the concatenation of the paths C and C' form a Hamiltonian cycle. ∎

To treat the general case, we need some notations. Let $n > k > l \geq 0$. We will denote the set of all k-subsets of $A_n = \{1,2,\ldots,n\}$ by $X_{n,k}$. For all $\alpha = \{x_1,\ldots,x_k\} \in X_{n,k}$, we will write $\min(\alpha) = \min\limits_{1 \leq i \leq k} x_i$.

We will also write $Y_{n,k} = \{\alpha \in X_{n,k} \mid \alpha \subseteq [n-k,n]\}$.

Now, for any choice π of representatives of k-subsets of A_n, $X_{n,k}(\pi)$ will denote the set of all ordered k-tuples of $\{1,2,\ldots,n\}$ which are ordered according to π, and $Y_{n,k}(\pi)$ is defined similarly. Note that $|X_{n,k}(\pi)| = \binom{n}{k}$ and $|Y_{n,k}(\pi)| = k+1$. Since $X_{n,k}(\pi)$ is the set of vertices of $C_{n,k,l}(\pi)$ and $Y_{n,k}(\pi) \subset X_{n,k}(\pi)$, then we can consider the full subgraph of $C_{n,k,l}(\pi)$ on the subset $Y_{n,k}(\pi)$.

Let us notice that the set $Y_{n,k}$ contains $k+1$ elements, namely, the k-tuples $\omega_1,\omega_2,\ldots,\omega_{k+1}$ such that $\omega_i = \{n-k,\ldots,n\}\backslash\{n-k-1+i\}$, for all $1 \leq i \leq k+1$. We need the following easy lemma.

Lemma 5.12 *Let $k \geq 4, n > k \geq 2l > 0$, and let $\alpha_1,\alpha_2 \in X_{n,k}\backslash Y_{n,k}$ such that $\alpha_1 \neq \alpha_2, \alpha_1 \subset [n-k-1,n], \{n-l+1,\ldots,n\} \subset \alpha_2$. Also, let π' be a choice of representatives on $X_{n,k}\backslash Y_{n,k}$ such that*

$$\pi'(\alpha_2) = (1,\ldots,l,c_1,\ldots,c_{k-2l},n-l+1,\ldots,n).$$

Then there exists an extension π of π' to the whole set $X_{n,k}$ such that, in the full subgraph of $C_{n,k,l}(\pi)$ on $Y_{n,k} \sqcup \{\alpha_1,\alpha_2\}$, there exists a Hamiltonian path which starts at α_1 and ends at α_2.

Proof. For $k \in \{4,5,6\}$ the claim can be checked directly, and we leave it to the reader as an exercise (notice that the essence of the claim depends on k and l only, and not on n, so there are finitely many (a small number of) cases to check). Let $k \geq 7$. Then $k - l \geq 4$.

Let also $A = \{\omega_k,\omega_{k+1}\}, B = \{\omega_{l+1},\ldots,\omega_{k-1}\}, C = \{\omega_1,\ldots,\omega_l\}$. We can define the representatives of ω_k and ω_{k+1} (thus extending π' to the set $(X_{n,k}\backslash Y_{n,k}) \cup A$; we still denote this extension by π') such that the following conditions hold:

(i) $\pi'(\alpha_1)$ and $\pi'(\omega_{k+1})$ agree at exactly l entries;

(ii) $\pi'(\omega_{k+1})$ and $\pi'(\omega_k)$ agree at exactly l entries;

(iii) $\pi'(\omega_k) = (n-k,n-k+1,\ldots,n-k+l-1,a_1,\ldots,a_{k-l})$ for some $a_1,\ldots,a_{k-l} \in \{n-k+l,\ldots,n\}$.

Then, since $k - l \geq 4$, by Lemma 5.7, we can define a choice of representatives on the set B (thus extending π' to $(X_{n,k}\backslash Y_{n,k}) \cup A \cup B$) such that for all $j \in \{l+1,\ldots,k-1\}$, the ordered k-tuple $\pi'(\omega_j)$ is in the form $(n-k, n-k+1, \ldots, n-k+l-1, x_1, \ldots, x_{k-l})$, moreover, the ordered k-tuples $\pi'(\omega_k), \ldots, \pi'(\omega_l)$ agree at the first l entries, and consecutive k-tuples in this sequence differ at all other $k-l$ entries.

Then we can define π' on the k-subset ω_l such that

$$\pi'(\omega_l) = (b_1, \ldots, b_{k-l}, n-l+1, \ldots, n)$$

for some $b_1, \ldots, b_{k-l} \in \{1, \ldots, n-l\}$, and $\pi'(\omega_{l+1}), \pi'(\omega_l)$ agree at exactly l entries.

Finally, again by Lemma 5.7, we can extend π' to $(X_{n,k} \backslash Y_{n,k}) \cup A \cup B \cup C$ (we will denote this final extension by π) such that for all $j \in \{1, \ldots, l\}$, the ordered k-tuple $\pi'(\omega_j)$ is in the form $(y_1, \ldots, y_{k-l}, n-l+1, \ldots, n)$, moreover, the ordered k-tuples $\pi'(\omega_l), \ldots, \pi'(\omega_1), \alpha_2$ agree at the last l entries, and any two consecutive k-tuples in this sequence differ at all other entries.

Thus we have obtained a Hamiltonian path in the full subgraph of $C_{n,k,l}(\pi)$ on $Y_{n,k} \cup \{\alpha_1, \alpha_2\}$ connecting α_1 to α_2. ■

Remark 5.13 Notice that, when $n = k+1$, we have $X_{n,k} = Y_{n,k}$. Then from the proof we also obtain the claim of Theorem 5.14 for the case of $n = k+1$.

Now we are ready to prove the main theorem of this section.

Theorem 5.14 *For all non-negative integers n, k and l, with $n \geq \max\{4, k+1\}$ and $k \geq \max\{2l, 1\}$, there is a choice of representatives π such that $C_{n,k,l}(\pi)$ is Hamiltonian.*

Proof. By Proposition 5.10 and Proposition 5.11, we may assume that $k \geq 4$ and $l \geq 1$.

Let $Z_{n,k,l} = \{\{i_1, \ldots, i_l\} \mid 1 \leq i_1 < i_2 < \ldots < i_l \leq n-k+l\}$, i.e. $Z_{n,k,l}$ is the set of all l-subsets of the set $\{1, 2, \ldots, n-k+l\}$ where we list the elements in increasing order. Note that $Z_{n,k,l}$ has exactly $r = \binom{n-k+l}{l}$ elements.

Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be the sequence of all elements of $Z_{n,k,l}$ such that the following conditions are satisfied:

(i) $\alpha_1 = \{1, 2, \ldots, l\}$,

(ii) $\alpha_r = \{n-k+1, n-k+2, \ldots, n-k+l\}$,

(iii) if $1 \leq i < j \leq r$ then $\min(\alpha_i) \leq \min(\alpha_j)$,

(iv) for all $m \in \{1, 2, \ldots, r-1\}$, $|\alpha_{m+1} \backslash \alpha_m| = 1$,

For every $\alpha = \{i_1, \ldots, i_l\} \in Z_{n,k,l}$ we define a set

$$V_\alpha = \{\{i_1, \ldots, i_l, j_1, \ldots, j_{k-l}\} \mid i_l < j_1 < \ldots < j_{k-l} \leq n\}$$

For any choice of representatives π which fixes $i_1, \ldots, i_l$, the set V_α can also be viewed as a subset of vertices of $C_{n,k,l}(\pi)$ so we will denote the full subgraph of $C_{n,k,l}(\pi)$ on V_α by $V_\alpha(\pi)$. Notice also that for any such π, the full subgraph of $C_{n,k,l}(\pi)$ on V_α is isomorphic to $C_{n-i_l,k-l}(\pi)$. Hence, by Proposition 5.10a), $V_\alpha(\pi)$ is Hamiltonian for any π which fixes $i_1, \ldots, i_l$.

We consider the sequence $V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_r}$ of mutually disjoint subsets. Notice that, for any choice of π, the subsets $V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_r}$ form a partition of the set of vertices of $C_{n,k,l}(\pi)$.

Let $r' = \min\ \{j \in \{1, 2, \ldots, r\} \mid \alpha_j \in Y_{n,k}\}$. Notice that by condition (iii) we have $\alpha_{r'-1} \subset [n-k-1, n]$. The idea of the proof is to use Proposition 5.10 and Lemma 5.12 to find choices of representatives on the subsets $V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_{r'}}$ and $\sqcup_{j \geq r'} V_{\alpha_j}$ such that the full subgraph on each of these subsets is Hamiltonian. Of course, we still need to match the ends of these paths to obtain a global Hamiltonian cycle on the whole graph.

We will treat V_{α_1} specially. Let $\gamma = (1, 2, \ldots, l, n-l+1, n-l+2, \ldots, n, c_1, \ldots, c_{k-2l})$, $\delta = (1, 2, \ldots, k)$ (so γ is an ordered k-tuple formed by $2l$ elements $1, 2, \ldots, l, n-l+1, n-l+2, \ldots, n$ and some remaining elements $c_1, \ldots, c_{k-2l}$; notice that for the existence of such γ we need the condition $k \geq 2l$). Let also

$$[\gamma] = \{1, 2, \ldots, l, n-l+1, n-l+2, \ldots, n, c_1, \ldots, c_{k-2l}\}, [\delta] = \{1, 2, \ldots, k\}$$

Note that $[\gamma], [\delta] \in V_{\alpha_1}$. By Proposition 5.10, there exists a choice of representatives π_1 on V_{α_1}, and a Hamiltonian path in $V_{\alpha_1}(\pi_1)$ which starts at $[\gamma]$ and ends at $[\delta]$, moreover, we can make π_1 agree with the ordering of γ and δ, i.e. $\pi_1([\gamma]) = \gamma$ and $\pi_1([\delta]) = \delta$. Let R_1 be such a path.

We will define the paths $R_2, R_3, \ldots$ inductively. For all $2 \leq i \leq r'-1$, by Proposition 5.10b) and conditions (iii)-(iv), there exists a Hamiltonian path R_i in $V_{\alpha_i}(\pi_i)$ for some choice of representatives π_i on V_{α_i} such that the beginning of the path R_i is connected to the end of the Hamiltonian path R_{i-1} in $V_{\alpha_{i-1}}(\pi_{i-1})$.

Let z be the end of the Hamiltonian path in $V_{\alpha_{r'-1}}(\pi_{r'-1})$. We have $\alpha_j \in Y_{n,k}$ for all $r' \leq j \leq r$. Let us also emphasize that we have already defined the choice of representatives on all $V_{\alpha_i}, 1 \leq i \leq r'-1$, so it remains to define it on $\sqcup_{j \geq r'} V_{\alpha_j}$.

By Lemma 5.12, there exists a choice of representatives $\pi_{r'}$ on $\sqcup_{j \geq r'} V_{\alpha_j}$ such that the full subgraph on $\sqcup_{j \geq r'} V_{\alpha_j}(\pi_{r'})$ contains a Hamiltonian path Q starting at z_1 and ending at z_2 such that z_1 is connected to $\pi_{r'-1}(z)$ and z_2 is connected to γ. Then $R_1 R_2 R_3 \ldots R_{r'-1} Q$ is a Hamiltonian cycle in $C_{n,k,l}(\pi)$ where the π is the unique extension of the choices of representatives $\pi_i, 1 \leq i \leq r'$. ■

Remark 5.15 When $l = k$, a $C_{n,k,l}$-graph is always disconnected unless $n = 1$ in which case the graph is trivial. When $n = k$, a $C_{n,k,l}$-graph is always trivial.

5.3. *The graph J_n contains a Hamiltonian path for all $n \geq 6$.*

THEOREM 5.16 *The graph J_n contains a Hamiltonian path for all $n \geq 6$.*

Proof. It is easy to verify the claim for $n \leq 14$ by direct checking (in the hardest case $n = 14$ we have the following Hamiltonian path in the graph J_{14}: $(1,6,7), (1,5,8), (2,5,7), (3,5,6), (3,4,7), (2,4,8), (2,3,9), (1,4,9), (1,3,10),$ $(1,2,11))$, so we will assume that $n \geq 15$.

For all $1 \leq i \leq [\frac{n}{3}]-1$, we define $s(i,n) = [\frac{n-i-1}{2}] - i$, and for all $i+1 \leq j \leq s(i,n)$ we define $a_j^{(i,n)} = (i, j, n-i-j)$.

For all $1 \leq i \leq [\frac{n}{3}]-1$, let also $L_i(n) = (a_{i+1}^{(i,n)}, \ldots, a_j^{(i,n)}, \ldots, a_{s(i,n)}^{(i,n)})$ and $R_i(n) = (a_{i+2}^{(i,n)}, \ldots, a_j^{(i,n)}, \ldots, a_{s(i,n)}^{(i,n)})$. (Notice that $L_i(n) = ((i, i+1, n-2i-1), (i, i+2, n-2i-2), \ldots, (i, j, n-i-j), \ldots, (i, [\frac{n-i-1}{2}], n-i-[\frac{n-i-1}{2}]))$, moreover, $L_i(n)$ and $R_i(n)$ are paths in the graph J_n). Notice that the path $L_i(n)$ contains all vertices of J_n of the form $(i, x, y), i < x < y$, and the path $R_i(n)$ is obtained from $L_i(n)$ by deleting the first vertex of it.

Also, for all $1 \leq i \leq [\frac{n}{3}]-2$, we define $S_i(n) = (a_{s(i,n)}^{(i,n)}, \ldots, a_{i+3}^{(i,n)}, a_{i+1}^{(i+1,n)}, a_{i+2}^{(i,n)}, a_{i+1}^{(i,n)})$. Let us make a very useful observation that $S_i(n)$ is a path in J_n.

Since $n \geq 15$, we have $[\frac{n}{3}]-1 \geq 4$ so there are at least 4 paths $L_i(n)$ in the graph J_n. (See Fig. 14)

The paths $L_i(n), 1 \leq i \leq [\frac{n}{3}]-1$ are mutually disjoint, and every vertex of the graph J_n

belongs to precisely one of these paths. Moreover, if $1 \leq i \leq [\frac{n}{3}]-2$ then the last vertex of $L_i(n)$ is connected to the last vertex of $L_{i+1}(n)$ by an edge. Then, for all $1 \leq i \leq [\frac{n}{3}]-2$, $R_{i+1}S_i$ is a path in J_n. (Let us also observe that if R_{i+1} contains at least three elements, then the path $R_{i+1}S_i$ covers exactly the set of vertices covered by the path $L_{i+1}L_i^{-1}$.) Furthermore, the first vertex of every $L_i(n)$ (for $i \geq 2$) is connected to the second and third vertices (if they exist) of $L_{i-1}(n)$. Then $S_{i+1}R_i$ is a path in J_n.

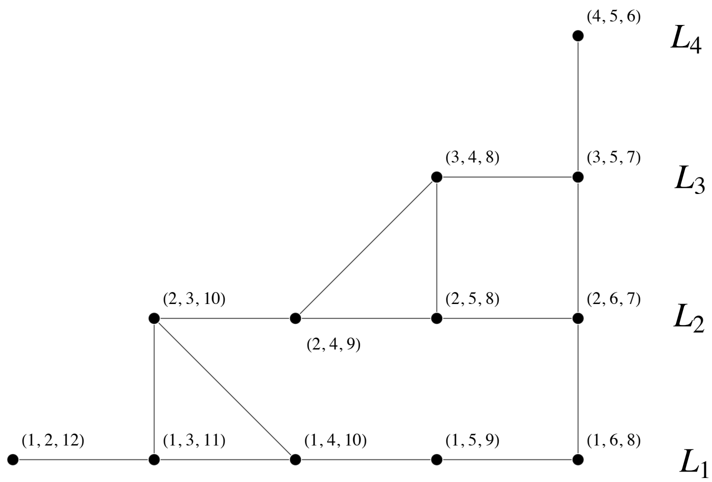

Figure 14. The 1st vertex of every $L_i(n)$ (for $i \geq 2$) is connected to the 2nd and 3rd vertices (if they exist) of $L_{i-1}(n)$ (note that not all edges of J_{15} are drawn in the picture)

The idea of the proof is to build a Hamiltonian path by following roughly the paths $L_1(n), L_2(n), \ldots, L_{s(i,n)}(n)$. The problem is that while the last vertex of $L_i(n)$ is connected to the last vertex of $L_{i+1}(n)$ by an edge, the first vertex of $L_i(n)$ is not connected to the first vertex of $L_{i+1}(n)$; hence $L_1(n)L_2(n)^{-1}L_3(n)L_4(n)^{-1}\ldots$ is not necessarily a path in J_n. Using the observations in the previous paragraph (i.e. using the paths $R_{i+1}S_i$ and $S_{i+1}R_i$), we will slightly tweak these paths $L_1(n), \ldots, L_{s(i,n)}(n)$ to obtain a Hamiltonian path. One of the issues in implementing this plan is the fact that the paths R_i and S_i could be degenerate, i.e. they may have less than three elements for certain values of i close to $\frac{n}{3}$.

Let $n = 6k + r, 0 \leq r \leq 5$. We will divide the proof into 4 cases. In each case we will present a Hamiltonian path in J_n, and in each case, the Hamiltonian path will end at the vertex $(1, 2, n-3)$.

Case 1: $r = 0$.
Let $L'(n) = ((2k-1, 2k, 2k+1), (2k-2, 2k, 2k+2))$. Then $L'(n)S_{2k-3}R_{2k-4}\ldots S_3R_2S_1$ is a Hamiltonian path in J_n.

Case 2: $r = 1$.
Then $L_{2k-1}R_{2k-2}S_{2k-3}R_{2k-4}S_{2k-5}\ldots R_2S_1$ is a Hamiltonian path.

Case 3: $r = 2$.
Then $L_{2k-1}^{-1}R_{2k-2}S_{2k-3}R_{2k-4}S_{2k-5}\ldots R_2S_1$ is a Hamiltonian path. (Notice that in this case the path L_{2k-1} contains two vertices).

Case 4: $3 \leq r \leq 5$.
In these cases, the path $L_{2k}L_{2k-1}^{-1}R_{2k-2}S_{2k-3}R_{2k-4}S_{2k-5}\ldots R_2S_1$ is Hamiltonian. (Notice that, when $r = 5$, the path L_{2k} consists of two vertices). ■

Remark 5.17 The graphs J_6 and J_7 are trivial, i.e. each has only one vertex and no edges. But for $n \geq 8$, J_n has at least two vertices and contains a vertex which is connected to only one other vertex therefore it is not Hamiltonian. Indeed, the vertex $(1, 2, n-3)$ is adjacent only to the vertex $(1, 3, n-4)$.

5.4. *The Hamiltonicity of P_n for $n \geq 8$.*

THEOREM 5.18 *The graph P_n is Hamiltonian for all $n \geq 8$.*

The proof is by induction on n. For the base of the induction, we observe that the graphs $P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}$ are Hamiltonian with a Hamiltonian cycle containing the special edge $((1, 2), (1, 3))$. (See Fig. 15 for the Hamiltonian cycles in the graphs P_8, P_9, P_{10}, P_{11} containing the special edge; we leave it to the reader as an exercise to find such cycles also in P_{12} and P_{13}).

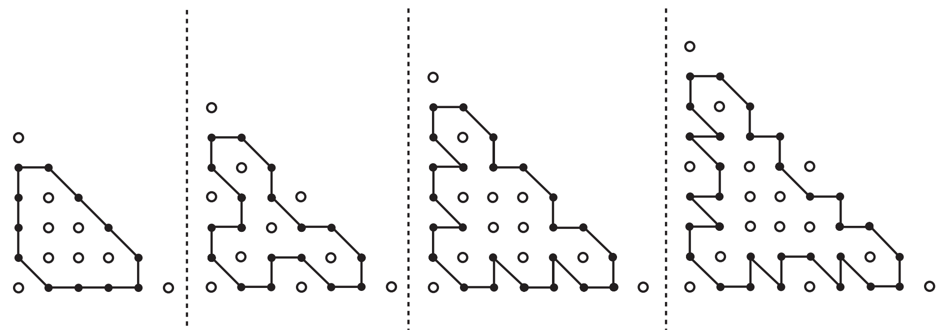

Figure 15. Hamiltonian cycles of P_8, P_9, P_{10} and P_{11} (empty nodes are points on the plane that do not belong to the graph)

Then, for the step of the induction, it suffices to prove the following.

PROPOSITION 5.19 *If P_n is Hamiltonian for $n \geq 8$ with a Hamiltonian cycle including the special edge, then so is P_{n+6}.*

Proof. For $n \geq 8$, let $\alpha_n = (x_1, \ldots, x_N)$ be a Hamiltonian cycle of P_n which includes the special edge, i.e. let (x_N, x_1) be the special edge of P_n.

Let $D_n = \{(x, y) \in \tau_{n+6} \mid x \geq 3, y \geq 3, ((n+6) - (x+y)) \geq 3\}$. We observe that

$$(x, y) \in D_n \Leftrightarrow (x-2, y-2) \in \tau_n$$

Moreover, for the vertices $(x_1, y_1), (x_2, y_2)$ in D_n,

$$((x_1, y_1), (x_2, y_2)) \in E(P_{n+6}) \Leftrightarrow ((x_1 - 2, y_1 - 2), (x_2 - 2, y_2 - 2)) \in E(P_n)$$

where $E(P_k)$ denotes the set of edges of P_k.

Thus the full subgraph of P_{n+6} on D_n is isomorphic to P_n where the isomorphism is given by mapping the vertex (x, y, z) to $(x-2, y-2, z-2)$.

Now, let $A_n = \{(x, y) \in \tau_{n+6} \mid 1 \leq x \leq 2, y > x\}, B_n = \{(x, y) \in \tau_{n+6} \mid 1 \leq y \leq 2, y < x\}, C_n = \{(x, y) \in \tau_{n+6} \mid 1 \leq n - (x+y) \leq 2\}$. Notice that A_n, B_n, C_n are subsets of the set of vertices of P_{n+6}; moreover, $A_n \cap B_n = \emptyset, A_n \cap C_n = \{(2, n-3)\}, B_n \cap C_n =$

$\{(n-3,2)\}$, $(A_n \cup B_n \cup C_n) \cap D_n = \emptyset$. Also, if n is even (the case of odd n is similar) by letting $n = 2m$ we observe that

$$\{(x,y) \in \Delta_{n+6} \mid 1 \le x \le 2\} \backslash A_n = \{(1,1),(2,2),(2,m-1),(2,n-4),(1,n-2)\},$$

$$\{(x,y) \in \Delta_{n+6} \mid 1 \le y \le 2\} \backslash B_n = \{(1,1),(2,2),(m-1,2),(n-4,2),(n-2,1)\},$$

$$\{(x,y) \in \Delta_{n+6} \mid 1 \le n-(x+y) \le 2\} \backslash C_n = \{(1,n-1),(n-1,1),(2,n-4),(n-4,2),(m-1,m-1)\}.$$

After these observations, it is easy to see that, for $n \ge 8$, the full subgraph of P_{n+6} on A_n has a Hamiltonian path β_1 which starts at $(2, n-3)$, ends at $(1,2)$, and passes through the edge $((2,5),(2,4))$. Let β_2 be the reflection of β_1 with respect to the line $y = x$. Then β_2 is a Hamiltonian path of the full subgraph of P_{n+6} on B_n which starts at $(n-3, 2)$ and ends at $(2,1)$.

It is also easy to see that the full subgraph of P_{n+6} on C_n has a Hamiltonian path which starts at $(n-3,2)$ and ends at $(2,n-3)$. Let us denote this path by β_3. Then $\beta = \beta_3\beta_1\beta_2^{-1}$ is a Hamiltonian cycle of the full subgraph of P_{n+6} on $A_n \cup B_n \cup C_n$.

Let β' be the cyclic shift of the cycle β which starts at $(2,5)$. Let also α' be the Hamiltonian cycle of the full subgraph of P_{n+6} on D_n which is the isomorphic image of α_n. Then $((2,5),(2,4))$ is an edge of β', and $((3,5),(3,4))$ is an edge of α'; notice that the edge $((3,5),(3,4))$ is the isomorphic image of the special edge $((1,3),(1,2))$. Since $((2,5),(3,5)), ((2,4),(3,4))$ are edges of P_{n+6} we immediately obtain a Hamiltonian cycle in P_{n+6}. ■

Remark 5.20 The graph P_6 is isomorphic to the cycle C_6 therefore it is also Hamiltonian. However, the graph P_7 is not Hamiltonian, in fact, it is disconnected.

Remark 5.21 The proof of Proposition 5.19 can be summarized as follows: we obtain a partition $(A_n \cup B_n \cup C_n) \sqcup D_n$ of the set of vertices of P_{n+8}. The union $A_n \cup B_n \cup C_n$ also is almost a partition in the sense that $A_n \cap B_n = \emptyset$, while the intersections $A_n \cap C_n$ and $B_n \cap C_n$ consist of a single point. The sets A_n and B_n are symmetric with respect to the $y = x$ line, while the full subgraph on the set D_n is isomorphic to the graph P_n. It is easy to find Hamiltonian paths in the full subgraphs on A_n, C_n, B_n such that, together, these paths form a cycle β. On the other hand, by the hypothesis, there exists a Hamiltonian cycle α' in the full subgraph on D_n. Furthermore, we observe that the cycle β contains consecutive vertices $u_1 = (2,5), u_2 = (3,5)$ and the cycle α' contains consecutive vertices $v_1 = (2,4), v_2 = (3,4)$ such that (u_1, v_1) and (u_2, v_2) are edges in the graph P_{n+6}. Thus we obtain a global Hamiltonian cycle.

6. CONCLUSION

We have explored the Hamiltonicity for $T_{n,k,l}$, $C_{n,k,l}$, J_n, and P_n. Still, there are questions that warrant further research such as whether our approach to $C_{n,k,l}$ may indeed impact the problem of Hamiltonicity of Johnson Graphs.

As mentioned at the start, our primary focus in this research was simply to show the circumstances in which Hamiltonicity holds in the abstract cases of our examples. In many cases, it is not trivial to actually find an instance of the path. It is known that determining the Hamiltonicity of arbitrary graphs is an NP-complete problem. We leave open the potential connection between the mathematical proofs and the applied examples as well as whether or not there exists optimal algorithms for particular cases (specifically where full pieces were generated such as Johnson's *Trio* and the second author's *maximum change*). For example, the final morphology for Johnson's *Trio* was painstakingly determined by hand. When beginning to write the piece, Johnson may not have even known whether or not such a construction was possible. Due to this, exactly how his solution relates to ours is not completely explored. For the case of *maximum change*, the rather simple solution used to create the piece was found independent of the proof. The solution generates a list for each of the 4 pitches that determines which instrument it is assigned over time by an algorithm that works as follows. For each pitch generate a permutation of (1,2,3,4). For the first pitch, simply repeat that (ordered) set 64 times. For the second pitch iterate the set 64 times, rotating it left by one position every iteration. For the third pitch, iterate the set 64 times, rotating it left by one position every 4 iterations. And finally, for the last pitch, iterate the set 64 times, rotating it left by one position every 16 iterations. Something similar to this solution should likely work for any $T_{n,n}$ and like the previous example, we have not fully explored the relationship between the proof and the algorithm used to generate the piece.

Finally, it could be that multiple paths satisfying our definitions are quite different phenomenologically. Despite the fact that most of our examples can be related to ideas of neo-Riemannian analysis, we have not investigated whether our abstractions (not at all based on tonal harmony) can be applied to earlier literature beyond the examples that instigated our research. There are works between the late romantic era and our examples that explore new ideas of voice leading particularly with respect to spectral morphing/modulation/interpolation. A notable example is Gérald Grisey's *Modulations* (1976–77) for orchestra where the formal framework of the piece is based largely on modulations from harmonic spectra to subharmonic spectra (see Rose 1996). Another set of apt example is Larry Polansky's *Psaltery* (1979), *Horn* (1989; rev. 1992), and *FreeHorn* (2004). In these pieces, Polansky interpolates/morphs between several sets of harmonic series by defining an algorithm that successively replaces harmonics in the source series by harmonics in the target series (see Fig. 16). While these examples relate more generally to new techniques of voice leading and not necessarily directly to our constructions, because they are well-defined (at least the Polansky examples are), they could be subjected to deeper mathematical investigations.

Regardless, J_n and P_n can clearly be seen as an extension of parsimonious voice leading and Cohn's notion of maximally smooth cycles to music that is not necessarily tonal or at least employs a more extended tonality where the whole gamut of chords with 3 pitches is used as opposed to just chords comprised of certain intervals such as thirds and perfect fifths. We also have not further abstracted this idea to chords with more than 3 pitches. An interesting, open problem is the Hamiltonicity of an extended version of J_n and P_n where the chord could be comprised of more than 3 pitches and some number of them stay the same while the others satisfy some sort of parsimonious voice leading constraint.

Ultimately, we feel that investigating the character of the Hamiltonian cycles of the graphs we define (and their possible extensions) with respect to other factors such as harmonic trajectory and algorithmic complexity may reveal paths that have different local and global properties of interest to musicians and mathematicians alike.

6.1. *Acknowledgements*

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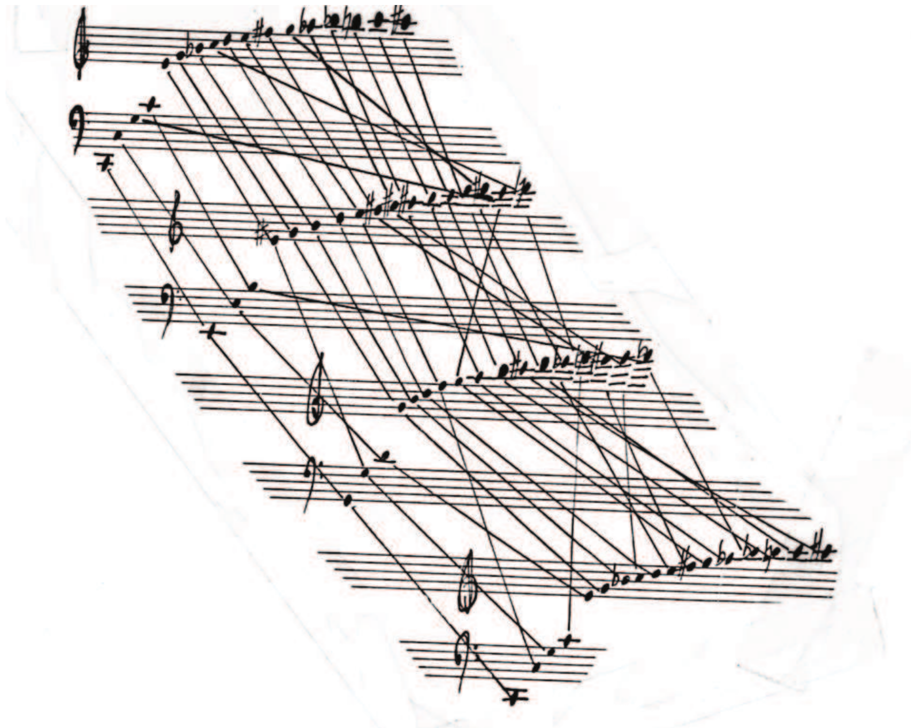

Figure 16. Harmonic trajectory of Polansky's *Psaltery* that shows how harmonics are replaced from each harmonic series to the next

by a referee which helped us to simplify the proof of Theorem 5.5 and correct a flawed argument in the proof of Lemma 5.2. We also would like to extend our gratitude to Thomas Fiore for so many helpful comments.

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