On dense subgroups of $\mathrm{Homeo}_+(I)$

Azer Akhmedov

Abstract: We prove that a dense subgroup of $\mathrm{Homeo}_+(I)$ is not elementary amenable and (if it is finitely generated) has infinite girth. We also show that the topological group $\mathrm{Homeo}_+(I)$ does not satisfy the *Stability of the Generators Property*, moreover, any finitely subgroup of $\mathrm{Homeo}_+(I)$ admits a faithful discrete representation in it.

1. Introduction

In this paper, we study basic questions about dense subgroups of $\mathrm{Homeo}_+(I)$ - the group of orientation preserving homeomorphisms of the closed interval $I = [0, 1]$ - with its natural C_0 metric. The paper can be viewed as a continuation of **[A1]** and **[A2]** both of which are devoted to the study of discrete subgroups of $\mathrm{Diff}_+(I)$.

Dense subgroups of connected Lie groups have been studied extensively in the past several decades; we refer the reader to **[BG1], [BG2], [Co], [W1], [W2]** for some of the most recent developments. A dense subgroup of a Lie group may capture the algebraic and geometric content of the ambient group quite strongly. This capturing may not be as direct as in the case of lattices (discrete subgroups of finite covolume), but it can still lead to deep results. It is often interesting if a given Lie group contains a finitely generated dense subgroup with a certain property. For example, finding dense subgroups with property (T) in the Lie group $SO(n+1, \mathbb{R}), n \geq 4$ by G.Margulis **[M]** and D.Sullivan **[S]**, combined with the earlier result of Rosenblatt **[Ro]**, led to the brilliant solution of the Banach-Ruziewicz Problem for $\mathbb{S}^n, n \geq 4$.

A major property that we are interested in for dense subgroups of $\mathrm{Homeo}_+(I)$ is not property (T) but amenability (incidentally, it is not known if $\mathrm{Homeo}_+(I)$ has a non-trivial subgroup with property (T)). A very natural example of a finitely generated dense subgroup of $\mathrm{Homeo}_+(I)$ is R.Thompson's group F in its standard representation in $\mathrm{PL}_+(I)$. The question about its amenability has been very popular in the last four decades. On the other hand, density of a finitely generated group in a large group $\mathrm{Homeo}_+(I)$ seems to be in conflict with the amenability. We prove the following theorem.

Theorem 1.1. *An elementary amenable subgroup of* $\mathrm{Homeo}_+(I)$ *cannot be dense.*

We do not know how to prove a stronger result by removing the adjective *elementary* from the statement of the theorem. Nevertheless, we expand on some of the ideas of the proof to obtain another fact about dense subgroups.

Theorem 1.2. *A finitely generated dense subgroup of* $\mathrm{Homeo}_+(I)$ *has infinite girth.*

As a corollary of this theorem we obtain that $girth(F) = \infty$ reproving the results from [**AST**], [**Br**] and [**A6**]. It follows from either of Theorem 1.1 and Theorem 1.2 that solvable groups cannot be dense in $\mathrm{Homeo}_+(I)$.

The notion of girth for a finitely generated group was first introduced in [**S**] in connection with the study of Heegaard splittings of closed 3-manifolds.

Definition 1.3. Let Γ be a finitely generated group. For any finite generating set S of Γ, $girth(\Gamma, S)$ will denote the minimal length of relations among the elements of S. Then we set

$$girth(\Gamma) = \sup_{\langle S\rangle=\Gamma, |S|<\infty} girth(\Gamma, S)$$

Basic properties of girth have been studied in [**A4**]. By definition above, an infinite cyclic group has infinite girth, but this fact should be viewed as a degeneracy since (as remarked in [**A4**]) any other group satisfying a law has a finite girth. We refer the reader to [**BE**], [**Nak1**], [**Nak2**], [**Y**] for further studies of girth. It will be also clear from the proof of Theorem 1.2 that the claim holds for the group of orientation preserving homeomorphisms of an arbitrary manifold (with a C_0 metric), and even in a more general setting of suitable metric spaces.

Let us point out that the claims of both Theorem 1.1 and Theorem 1.2 hold for connected semi-simple real Lie groups. Indeed, it is proved in [**BG1**] that any dense subgroup of a connected real semi-simple Lie group G contains a non-abelian free subgroup hence it must be non-amenable. On the other hand, by the main result of [**A5**], any finitely generated linear group with a non-abelian free subgroup has infinite girth. Combining this result with Proposition 1 of [**A5**] it is not difficult to show that a finitely generated subgroup of a connected real Lie group with a non-abelian free subgroup has infinite girth.

In Section 2 of the paper, we discuss the so-called stability of the generator's property which also holds for simple Lie groups but we show that this property fails in $\mathrm{Homeo}_+(I)$.

We will say that a topological group G satisfies *Stability of the Generators Property* (SGP) if for any finitely generated dense subgroup Γ of G generated by elements $g_1, \dots, g_n$, there exists an open non-empty neighborhood U of the identity such that if $h_i \in g_i U, 1 \leq i \leq n$ then the group generated by $h_1, \dots, h_n$ is also dense.

For a topological group G, the SGP can be viewed as a stability of any finite generating set (in a topological sense: a subset $S \subseteq G$ generates G if it generates a dense subgroup in G). It is immediate to see that the group $\mathbb{R}$ does not satisfy SGP. On the other hand, it is not difficult to show the SGP for connected simple Lie groups, using Margulis-Zassenhaus Lemma. This lemma (discovered by H.Zassenhaus in 1938, and later rediscovered by G.Margulis in 1968) states that in a connected Lie group H there exists an open non-empty neighborhood U of the identity such that any discrete subgroup generated by elements from U is nilpotent (see [**Ra**]). For example, if H is a simple Lie group (such as $SL_2(\mathbb{R})$), and $\Gamma \leq H$ is a lattice, then Γ cannot be generated by elements too close to the identity. It is easy to see that (or see [**A2**] otherwise) the lemma fails for $\mathrm{Homeo}_+(I)$. We prove the following theorem.

Theorem 1.4. *The topological group* $\mathrm{Homeo}_+(I)$ *does not satisfy Stability of the Generators Property.*

We indeed prove more: given any finitely generated subgroup Γ of $\mathrm{Homeo}_+(I)$, and an arbitrary $\epsilon > 0$, we show that one can find an isomorphic copy Γ_1 of Γ generated by elements from an ϵ-neighborhood of the generators of Γ such that Γ_1 is discrete. This also shows that *any finitely generated subgroup of* $\mathrm{Homeo}_+(I)$ *admits a faithful discrete representation in it.*

We also prove that *every* finite generating set of $\mathrm{Homeo}_+(I)$ is indeed unstable. Furthermore, given any n-tuple $(g_1, \dots, g_n)$ generating *a dense subgroup*, one can find another n-tuple $(h_1, \dots, h_n)$ arbitrarily close to it which generates *a discrete subgroup.*

It is a well known fact (see [**G**] or [**Nav2**]) that any countable left-orderable group embeds in $\mathrm{Homeo}_+(I)$. We modify this argument slightly to obtain the claim of Theorem 1.4.

Let us emphasize that, despite the simplicity of the argument in [**G**], it does not produce a smooth embedding. Indeed, there are interesting examples of finitely generated left-orderable groups which do not embed in $\mathrm{Diff}_+(I)$ [**Be**, [**Nav1**]. For the group $\mathrm{Diff}_+(I)$, we do not know if the property SGP holds in either C_1 or C_0 metric; it is also unknown to us if every finitely generated subgroup $\Gamma \leq \mathrm{Diff}_+(I)$ admits a faithful C_1-discrete representation in $\mathrm{Diff}_+(I)$. Much worse, we even do not know if $\mathrm{Diff}_+(I)$ contains any finitely generated C_1-dense subgroup at all!

Acknowledgment: The question about the Stability of the Generators Property was brought to my attention by Viorel Nitica. It is a pleasure to thank him for a motivating conversation.

2. Instability of the Generators

In this section, we will prove Theorem 1.4. For $f \in C[0,1]$, $||f||$ will denote the usual C_0-norm, i.e. $||f|| = \max\limits_{x\in[0,1]} |f(x)|$.

First, we need the notion of a C_0-strongly discrete subgroup from [**A1**]:

Definition 2.1. A subgroup Γ is C_0-strongly discrete if there exists $\delta > 0$ and $x_0 \in (0,1)$ such that $|g(x_0) - x_0| > \delta$ for all $g \in \Gamma\backslash\{1\}$.

Notice that C_0-strongly discrete subgroups are C_0-discrete. The following theorem is stronger than Theorem 1.4.

Theorem 2.2. *Let* Γ *be a subgroup of* $\mathrm{Homeo}_+(I)$ *generated by finitely many homeomorphisms* $f_1, \ldots, f_s$*, and* $\epsilon > 0$*. Then there exist* $g_1, \ldots, g_s \in \mathrm{Homeo}_+(I)$ *such that* $\max\limits_{1\leq i\leq s} ||g_i - f_i|| < \epsilon$*, moreover, the subgroup* Γ_1 *generated by* $g_1, \ldots, g_s$ *is* C_0*-strongly discrete, and* Γ_1 *is isomorphic to* Γ.

Proof. Let $(x_0, x_1, \ldots)$ be a countable dense sequence in $(0,1)$ where $x_0 = \frac{1}{2}$, and let $\delta = \frac{1}{10}\min\{\epsilon, 1\}$. Since Γ is finitely generated it is countable and left-orderable with a left order $\prec$ such that for all $h_1, h_2 \in \Gamma$, we have $h_1 \prec h_2$ iff for some $n \geq 0, h_1(x_n) < h_2(x_n)$ and $h_1(x_i) = h_2(x_i)$ for all $i < n$.

Let $\gamma_0, \gamma_1, \gamma_2, \ldots$ be all elements of Γ where $\gamma_0 = 1$. We will build homeomorphisms $\eta_0, \eta_1, \eta_2, \ldots$ such that they generate a subgroup Γ_1 satisfying the following conditions:

(i) $\eta_0 = 1$;

(ii) there exists an isomorphism $\phi : \Gamma \to \Gamma_1$ such that $\phi(\gamma_n) = \eta_n$ for all $n \geq 0$;

(iii) $d_0(\gamma_n, \eta_n) < \epsilon$ for all $n \geq 1$;

(iv) $\eta_n(x_0) \notin (x_0 - \delta, x_0 + \delta)$ for all $n \geq 1$.

First, we define $\eta_n(x_0)$ inductively for all $n \geq 1$. We let $\eta_1(x_0)$ to be any number in $(0, 1)$ such that

(i) $\eta_1(x_0) \notin (x_0 - \delta, x_0 + \delta)$;

(ii) $|\eta_1(x_0) - \gamma_1(x_0)| < \frac{\epsilon}{2}$;

(iii) $(\eta_1(x_0) - x_0)(\gamma_1(x_0) - x_0) \geq 0$, i.e. $\eta_1(x_0)$ and $\gamma_1(x_0)$ are on the same side of x_0.

Now suppose $\eta_1(x_0), \dots, \eta_n(x_0)$ are defined. To define $\eta_{n+1}(x_0)$ we consider the following three cases.

Case 1: $\gamma_i \prec \gamma_{n+1}$ for all $0 \leq i \leq n$.

Then we let $\eta_{n+1}(x_0)$ be any number in $(\frac{1}{2}+\delta, 1)$ such that $\eta_{n+1}(x_0) > \eta_i(x_0), 1 \leq i \leq n$ and $|\eta_{n+1}(x_0) - \gamma_{n+1}(x_0)| < \frac{\epsilon}{2}$

Case 2: $\gamma_i \prec \gamma_{n+1} \prec \gamma_j$ for some $i, j \in \{1, \dots, n\}$ where for all $k \in \{1, \dots, n\} \backslash \{i, j\}$ either $\gamma_k \prec \gamma_i$ and $\gamma_j \prec \gamma_k$.

In this case, we let $\eta_{n+1}(x_0) \in (\eta_i(x_0), \eta_j(x_0))$ and $|\eta_{n+1}(x_0) - \gamma_{n+1}(x_0)| < \frac{\epsilon}{2}$.

Case 3: $\gamma_{n+1} \prec \gamma_i$ for all $0 \leq i \leq n$.

Then we let $\eta_{n+1}(x_0)$ be any number in $(0, \frac{1}{2}-\delta)$ such that $\eta_{n+1}(x_0) < \eta_i(x_0), 1 \leq i \leq n$ and $|\eta_{n+1}(x_0) - \gamma_{n+1}(x_0)| < \frac{\epsilon}{2}$.

Thus we have defined the orbit $O(x_0) = \{\eta_n(x_0) \mid n \geq 0\}$, and $\eta_n(x_0) \notin (x_0 - \delta, x_0 + \delta)$ for all $n \geq 1$. Then we can extend the definition of $\eta_n, n \geq 1$ to the whole $O(x_0)$ by setting $\eta_n(\eta_m(x_0)) = (\eta_n \eta_m(x_0))$ for all $m, n \geq 0$.

Now we extend the definitions of $\eta_n, n \geq 1$ to the set of all accumulation points of $O(x_0)$: let z be an accumulation point of $O(x_0)$, so $z = \lim\limits_{k \to \infty} z_k$ where $z_k = \eta_{n_k}(x_0), k \geq 1$. For all $n \geq 1$, we let $\eta_n(z) = \lim\limits_{k \to \infty} \eta(z_k)$.

Since the set $[0, 1] \backslash \overline{O(x_0)}$ is open, it is a union of countably many disjoint open intervals. Then we can extend the definition of the maps $\eta_n, n \geq 1$ affinely to the whole $[0, 1]$.

By construction, the group $\Gamma_1 = \{\eta_0, \eta_1, \ldots\}$ is isomorphic to Γ, moreover, $\eta_n(x_0) \notin (x_0 - \delta, x_0 + \delta)$ for all $n \geq 1$. Thus Γ_1 is C_0-strongly discrete. $\square$

Corollary 2.3. Any finitely generated subgroup of $\mathrm{Homeo}_+(I)$ admits a discrete embedding in it.

We do not know if the claim of the corollary holds for $\mathrm{Diff}_+(I)$ in C_1 metric. It is worth mentioning that not every finitely subgroup of a Lie group admits a discrete embedding in it: the group $\mathbb{Z} \wr \mathbb{Z}$ embeds in $GL(2, \mathbb{R})$ but does not embed discretely in any connected real Lie group.

3. Amenable subgroups of $\mathrm{Homeo}_+(I)$

In this section, we prove Theorem 1.1. First, we give a separate proof for solvable groups. The following proposition seems interesting independently.

Proposition 3.1. *Let $\Gamma \leq \mathrm{Homeo}_+(I)$ be a dense subgroup, and N be a non-trivial normal subgroup of Γ. Then N is dense.*

Proof. Let $\epsilon > 0$ and $\phi \in \mathrm{Homeo}_+(I)$. We can choose a natural number n and $a_1, \ldots, a_n, b_1, \ldots, b_n \in (0,1)$ such that $\frac{1}{n} < \frac{\epsilon}{2}, a_i = \frac{i}{n+1}, 0 \leq i \leq n+1, 0 = b_0 < b_1 < b_2 < \ldots < b_n < b_{n+1} = 1$, and the following three conditions hold:

(c1) $a_i \neq b_j$ for all $i, j \in \{1, \ldots, n\}$

(c2) $|b_i - \phi(a_i)| < \frac{\epsilon}{8}, 1 \leq i \leq n$.

(c3) $|b_{i+1} - b_i| < \frac{\epsilon}{8}, 0 \leq i \leq n$.

Let also $p = \min\{a_1, b_1\}, q = \max\{a_n, b_n\}$. Since Γ is dense, it does not have a global fixed point in $(0,1)$. Then, there exists $f \in N$ such that $f(\frac{p}{2}) > q + \frac{1-q}{2}$.

Let $c_k = f(a_k), d_k = f(b_k), 1 \leq i \leq n$ and $c_0 = d_0 = q, c_{n+1} = d_{n+1} = 1$. Then $q = c_0 < c_1 < \ldots < c_n < 1$, and $q = d_0 < d_1 < \ldots < d_n < 1$.

Let $\delta_0 = \frac{1}{16}\min\{\delta_1, \delta_2, \delta_3, \epsilon\}$ where

$$\delta_1 = \min_{0 \leq i \leq n} |c_{i+1} - c_i|, \ \delta_2 = \min_{0 \leq i,j \leq n} |a_i - b_j|, \ \delta_3 = \min\{p, 1-q\}$$

Then there exists a positive $\delta < \delta_0$ such that for all $k \in \{1, \ldots, n\}$, we have $f^{-1}(I_k) \subset (b_k - \delta_0, b_k + \delta_0)$ where $I_k = (d_k - \delta, d_k + \delta), 1 \leq k \leq n$.

Now, let $J_k = (b_k - \delta_0, b_k + \delta_0), L_k = (b_k - 2\delta_0, b_k + 2\delta_0), 1 \leq k \leq n$.

Notice that J_k is a subinterval of $L_k, 1 \leq k \leq n$, and the intervals $L_1, \dots, L_n, I_1, \dots, I_n$ are mutually disjoint. Moreover, all of the intervals $I_1, \dots, I_n$ lie on the right side of $q + \frac{1-q}{2}$ while all of the intervals $L_1, \dots, L_n$ lie on the left side of $q + \frac{1-q}{2}$.

By the density of Γ, we can find $g \in \Gamma$ such that for all $k \in \{1, \dots, n\}$ the following conditions hold:

(i) $g(c_k) \in I_k$;

(ii) $g^{-1}(J_k) \subset L_k$.

Then $g^{-1}f^{-1}gf(a_k) \in L_k$ for all $k \in \{1, \dots, n\}$. Then using conditions (c2) and (c3) we easily obtain that $||\phi - g^{-1}f^{-1}gf|| < \epsilon$. $\square$

We now observe an important corollary of Proposition 3.1.

Corollary 3.2. A solvable subgroup of $\mathrm{Homeo}_+(I)$ is not dense.

Proof. Indeed, if Γ is a solvable dense subgroup of $\mathrm{Homeo}_+(I)$ then it has a non-trivial normal Abelian subgroup N. By Proposition 3.1, N is dense. But since N is Abelian, it has a non-trivial cyclic normal subgroup C. Again, by Proposition 3.1, C is dense. However, a cyclic subgroup cannot be dense. Contradiction. $\square$

Remark 3.3. In fact, by a more direct argument one can show that an Abelian subgroup G of $\mathrm{Homeo}_+(I)$ cannot be $\frac{1}{8}$-dense, and a metabelian subgroup cannot be $\frac{1}{16}$-dense. (a subgroup G is called δ-*dense* if for all $f \in \mathrm{Homeo}_+(I)$, f lies in a distance δ apart from G.)

Our goal is now to extend the corollary to show that a dense subgroup of $\mathrm{Homeo}_+(I)$ cannot be elementary amenable.

For the convenience of the reader, let us recall that the class of amenable groups is closed under the following four natural processes of forming new groups out of the old ones: **(I)** subgroups, **(II)** quotients, **(III)** extensions, and **(IV)** direct unions. Following C.Chou [**C**], let us denote the class of Abelian groups and finite groups by EG_0. Assume that $\alpha > 0$ is an ordinal and we have defined EG_β for all ordinals $\beta < \alpha$. Then if α is a limit ordinal, set $\mathrm{EG}_\alpha = \bigsqcup_{\beta < \alpha} \mathrm{EG}_\beta$ and if α is not a limit ordinal, set EG_α is the class of groups which can be obtained from groups in $\mathrm{EG}_{\alpha-1}$ by either applying process **(III)** or process **(IV)** once and only once. It is proved that each class EG_α is closed under processes **(I)** and **(II)** and $\mathrm{EG} = \cup\{\mathrm{EG}_\alpha : \alpha \text{ is an ordinal } \}$ is the smallest class of groups which contain all finite and Abelian groups and is closed under the processes **(III)** and **(IV)**. A group from the class EG is called *elementary amenable group*. Some basic and interesting properties of these groups have been studied in [**C**].

A subgroup of $\mathrm{Homeo}_+(I)$ from class EG_0 is Abelian and Abelian groups are not dense by Corollary 3.2. Using this fact as a base of a transfinite induction, one would want to establish the step of it to prove that an elementary amenable subgroup is not dense. Assume that we can prove this claim for the groups of classes EG_β for all $\beta < \alpha$. If α is a limit ordinal then by definition of EG_α, any group Γ from it belongs to a class EG_β for some $\beta < \alpha$ thus we conclude by the inductive assumption that Γ is thin. If α is not a limit ordinal then there are two ways to obtain Γ from $\mathrm{EG}_{\alpha-1}$: **(i)** Γ is an extension of A by B where A, B are non-trivial subgroups from $\mathrm{EG}_{\alpha-1}$, and **(ii)** Γ is a direct union of $\{\Gamma_\tau\}, \Gamma_\tau \in \mathrm{EG}_{\alpha-1}$.

In Case (i), if Γ is dense then, by Proposition 3.1, the non-trivial normal subgroup B is also dense; but this contradicts the inductive assumption. However, in Case (ii), we are unable to carry out the step, for the following reason: a directed union of countably many nowhere dense subgroups of $\mathrm{Homeo}_+(I)$ can indeed be dense!

To overcome this difficulty, we would like to introduce a concept of *thin groups* which helps us to take care of the problem. For an integer n, let

$$\mathrm{sgn}(n) = \begin{cases} 1 & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ -1 & \text{if } n < 0 \end{cases}$$

Definition 3.4. Let $N \geq 1$ be an integer. A group Γ is called N-thin if for all $a, b \in \Gamma$ there exists a word $W(a,b) = a^{n_1} b^{n_2} \dots a^{n_{2k-1}} b^{n_{2k}} a^{n_{2k+1}}$ such that $W(a,b) = 1 \in \Gamma$ where $n_2, \dots, n_{2k}$ are non-zero, moreover, $\mathrm{sgn}(n_1) + \dots + \mathrm{sgn}(n_{2k+1}) = 0$, and $|\mathrm{sgn}(n_1) + \dots + \mathrm{sgn}(n_i)| \leq N$ for all $i \in \{1, \dots, 2k+1\}$.

In the above definition, the quantity $\max\limits_{1 \leq i \leq 2k+1} |\mathrm{sgn}(n_1) + \dots + \mathrm{sgn}(n_i)|$ will be called *the width* of the word $W(a,b)$, and the quantity $\max\limits_{1 \leq i \leq 2k+1} |n_i|$ will be called *the height* of the word $W(a,b)$. If $W(a, b_1, \dots, b_r)$ is a reduced word in $r+1$ letters such that it does not contain a subword of type $b_i b_j, b_i^{-1} b_j, b_i b_j^{-1}, b_i^{-1} b_j^{-1}, i \neq j$ then we define the width of W be equal to the width of $W(a, b, \dots, b)$. Similarly, we define the height of $W(a, b_1, \dots, b_r)$ to be the sum of absolute values of exponents of $b_j, 1 \le j \le r$.

Definition 3.5. A group is called thin if it is N-thin for some $N \ge 1$.

Let us observe the following important facts.

Proposition 3.6. *(i) a subgroup of an N-thin group is N-thin;*
(ii) a quotient of an N-thin group is N-thin;
(iii) an extension of an N-thin group by an M-thin group is $(M+N)$-thin;
(iv) a directed union of N-thin groups is N-thin;
(v) Abelian groups are 1-thin;
(vi) finite groups are 1-thin. $\square$

Thin groups turn out interesting from a pure group-theoretical point of view. Despite Proposition 3.6, not all elementary amenable groups are thin. Conversely, the class of thin groups includes interesting groups which are not elementary amenable (and not even amenable). Still, thin groups are useful in understanding the concept of amenability; here, we will limit ourselves to pointing out the following basic property of these groups.

Proposition 3.7. *A thin subgroup Γ of* $\mathrm{Homeo}_+(I)$ *is not dense.*

Proof. Let $n = 2N+2, 0 < a_1 < b_1 < a_2 < b_2 < \dots < a_n < b_n < 1$, and $x_0 \in (a_{N+1}, b_{N+1})$. Let also

$$S = \{0, x_0, 1\} \sqcup \{a_1, \dots, a_n\} \sqcup \{b_1, \dots, b_n\}$$
$$\text{and } \epsilon = \frac{1}{10} \min\{x - y \mid x, y \in S, x \neq y\}$$

We choose two homeomorphisms $f, g \in \mathrm{Homeo}_+(I)$ such that the following conditions hold:

(i) $Fix(f) = \{a_1, \dots, a_n\} \cup \{0, 1\}, Fix(g) = \{b_1, \dots, b_n\} \cup \{0, 1\}$;

(ii) for all $x \in I, f(x) \ge x$ and $g(x) \ge x$;

(iii) for all $x \in I$ if $\min\{x - z \mid z \in \{0,1\} \sqcup \{a_1, \dots, a_n\}\} > 2\epsilon$ then $f(x) - x > \epsilon$;

(iv) for all $x \in I$ if $\min\{x - z \mid z \in \{0,1\} \sqcup \{b_1, \dots, b_n\}\} > 2\epsilon$ then $g(x) - x > \epsilon$.

Since Γ is dense, we can choose $\phi, \psi \in \mathrm{Homeo}_+(I)$ such that $||\phi - f|| < \epsilon$ and $||\psi - g|| < \epsilon$. Then the following conditions hold:

(i) $Fix(\phi)\backslash\{0,1\} \subset \bigsqcup_{1\leq i\leq n}(a_i - 2\epsilon, a_i + 2\epsilon)$;

(ii) $\phi(x) > x$ for all $x \notin \bigsqcup_{1\leq i\leq n}(a_i - 2\epsilon, a_i + 2\epsilon) \sqcup [0, 2\epsilon) \sqcup (1 - 2\epsilon, 1]$

(ii) $Fix(\psi)\backslash\{0,1\} \subset \bigsqcup_{1\leq i\leq n}(b_i - 2\epsilon, b_i + 2\epsilon)$;

(iv) $\psi(x) > x$ for all $x \notin \bigsqcup_{1\leq i\leq n}(b_i - 2\epsilon, b_i + 2\epsilon) \sqcup [0, 2\epsilon) \sqcup (1 - 2\epsilon, 1]$

Notice that the intervals $[0, 2\epsilon], [a_1-2\epsilon, a_1+2\epsilon], [b_1-2\epsilon, b_1+2\epsilon], \dots, [a_n-2\epsilon, a_n + 2\epsilon], [b_n - 2\epsilon, b_n + 2\epsilon], [1 - 2\epsilon, 1]$ are mutually disjoint, and x_0 does not belong to any of them. Moreover, for a sufficiently big positive integer m, and for all $i \in \{2, \dots, n-1\}$, we have

$\phi^{-m}([b_i-2\epsilon, b_i+2\epsilon] \subset (a_i, a_i+2\epsilon), \phi^m([b_i-2\epsilon, b_i+2\epsilon] \subset (a_{i+1}-2\epsilon, a_{i+1})$

and

$\psi^{-m}([a_i-2\epsilon, a_i+2\epsilon] \subset (b_{i-1}, b_{i-1}+2\epsilon), \psi^m([a_i-2\epsilon, a_i+2\epsilon] \subset (b_i-2\epsilon, b_i)$

We also have

$$\phi^{-m}(x_0) \in (a_{N+1}, a_{N+1} + 2\epsilon), \phi^m(x_0) \in (a_{N+2} - 2\epsilon, a_{N+2}),$$

and

$$\psi^{-m}(x_0) \in (b_N, b_N + 2\epsilon), \psi^m(x_0) \in (b_{N+1} - 2\epsilon, b_{N+1})$$

Then we let $a = \phi^m, b = \psi^m$, and observe that for sufficiently big m, $W(f, g)(x_0) \in \bigsqcup_{1\leq i\leq n}(a_i-2\epsilon, a_i+2\epsilon)\sqcup \bigsqcup_{1\leq 1\leq n}(b_i-2\epsilon, b_i+2\epsilon)$ for all reduced words $W(a, b) = a^{n_1}b^{n_2}\dots a^{n_{2k-1}}b^{n_{2k}}a^{n_{2k+1}}$ where $|n_1 + \dots + n_i| \leq N$ for all $i \in \{1, \dots, 2k+1\}$. Hence $W(x_0) \neq x_0$, then $W \neq 1 \in \Gamma$. $\square$

Not every elementary amenable group is thin, thus we cannot apply Proposition 3.7 to prove Theorem 1.1. However, we make formulate a property for subgroups of $\mathrm{Homeo}_+(I)$ which will be a substitute for thinness. First, we need to introduce the notion of span for elements as well as for subgroups of $\mathrm{Homeo}_+(I)$.

Definition 3.8. For all $g \in \mathrm{Homeo}_+(I)$ we let

$$Span(g) = sup\{|J| \ : \ J \text{ is a subinterval of } (0,1), Fix(g) \cap J = \emptyset\}.$$

For subgroups $G \leq \mathrm{Homeo}_+(I)$ we let $Span(G) = \sup_{g\in G} Span(g)$.

Let us observe that if $G \leq \mathrm{Homeo}_+(I)$ and N is a normal subgroup of G with $Span(N) < Span(G)$ then there exists a subgroup $K \leq \mathrm{Homeo}_+(I)$, a normal subgroup $N_1 \trianglelefteq G$ and $\delta > 0$ such that $K \cong G/N_1$ and $Span(h) = Span(\phi(h))$ for all $h \in G$ with $Span(h) >$

$Span(G) - \delta$, where $\phi : G \to G/N_1 \cong K$ is the compostion of the quotient epimorphism with the isomorphism $G/N_1 \cong K$. This important observation allows us to make the following claim:

Proposition 3.9. *Let G be a countable elementary amenable group. Then there exists a sequence $\mathcal{H} = (h_k)_{k\geq 1}$ of elements of G such that the following conditions hold:*

(a1) if $Span(G) \leq \frac{1}{2}$ then the sequence $Span(h_n)$ is increasing and $\lim\limits_{n\to\infty} Span(h_n) = Span(G)$, but if $Span(G) > \frac{1}{2}$ then $Span(h_n) > \frac{1}{2}$ for all $n \geq 1$;

(a2) for all $g \in G, N \geq 1$ there exists some $r \geq 1, k_i \geq N, 1 \leq i \leq r$ and a non-trivial reduced word $W(x, y_1, \dots, y_r)$ of width at most two such that $Span(W(g^{m_0}, h_{k_1}^{m_1}, \dots, h_{k_r}^{m_r})) \leq \frac{1}{2}$ for all $m_i \geq 1, 0 \leq i \leq r$. $\square$

The proof is a straightforward check by a transfinite induction. In trying to apply this proposition (in proving, similarly to the proof of Proposition 3.7, that elementary amenable subgroups cannot be dense), one encounters the following problem: the sequence $(h_n)_{n\geq 1}$ may converge to identity or "tend to infinity" (the latter means that $N_1(h_n) \to \infty$; see Definition 3.10).

To overcome this difficulty, we will introduce a more subtle concept related to thinness. First, we need to introduce the notion of norm (for the elements of $\mathrm{Homeo}_+(I)$).

Definition 3.10. For all $g \in \mathrm{Homeo}_+(I)$, we let

$$N_1(g) = sup\{\frac{1}{\epsilon} \mid 0 < \epsilon < \frac{1}{2}, g(\epsilon) > 1 - \epsilon \text{ or } g(1-\epsilon) < \epsilon\},$$

$$N_2(g) = sup\{\frac{1}{\epsilon} \mid 0 < \epsilon < \frac{1}{2}, g(\frac{1}{2}) > 1 - \epsilon \text{ or } g(\frac{1}{2}) < \epsilon\},$$

$$N_3(g) = \frac{1}{|g(\frac{1}{2}) - \frac{1}{2}|}, N(g) = \max\{N_2(g), N_3(g)\}.$$

The quantity $N_1(g)$ measures how close the element is to infinity. $N_2(g)$ is a refinement of this notion which is more suitable for our purposes. Let us also clarify that in the case of $g(\frac{1}{2}) = \frac{1}{2}$ we have $N(g) = \infty$.

We now consider the following technical property for subgroups of $\mathrm{Homeo}_+(I)$ as a modification of thinness. We say a subgroup $G \leq \mathrm{Homeo}_+(I)$ is *quasi-thin* if there exists a sequence $\mathcal{H} = (h_n)_{n\geq 1}$ in G such that the following condtions hold:

(c1) if $Span(G) \leq \frac{1}{2}$ then the sequence $Span(h_n)$ is increasing and $\lim\limits_{n\to\infty} Span(h_n) = Span(G)$, but if $Span(G) > \frac{1}{2}$ then $Span(h_n) > \frac{1}{2}$ for all $n \geq 1$

(c2) if there exists $h \in G$ such that $Span(h) > \frac{1}{2}$ then for all $n \geq 1$, $N(h_n) \leq \max\{2N(h), 100\}$;

(c3) for all $g \in G, N \geq 1$ there exists some $r \geq 1, k_1, \ldots, k_r > N$ and a reduced word $W(x, y_1, \ldots, y_r)$ of height and width at most two such that $Span(W(g^n, h_{k_1}, \ldots, h_{k_r})) \leq \frac{1}{2}$ for all $n \geq 1$.

Let us emphasize that condition (c1) is identical to condition (a1), while condition (c3) is a modification of condition (a2).

Despite the technicalities of conditions (c1)-(c3), by a transfinite induction, it is straightforward to see that all countable elementary amenable subgroups of $\mathrm{Homeo}_+(I)$ are quasi-thin. Indeed, one can check that all three of these conditions are preserved under the operations **(III)** and **(IV)**. Thus it remains to prove the following

Proposition 3.11. *A quasi-thin group is not dense in* $\mathrm{Homeo}_+(I)$.

Proof. Let Γ be a quasi-thin dense subgroup. Then there exists $h \in \Gamma$ with $Span(h) > \frac{1}{2}$ therefore we have a sequence $\mathcal{H}$ of elements of Γ such that conditions (c1)-(c3) hold. Then there exists points p_0, p, q, q_0 such that the following conditions hold:

1) $0 < p_0 < p < \frac{1}{2} < q < q_0 < 1$,

2) $|p - q| > \frac{1}{2}$,

3) for all $k \geq 1$, $p_0 < h_k^{-1}(\frac{1}{2}) < p, \ q_0 < h_k(\frac{1}{2}) < q$.

Now, let $f \in \mathrm{Homeo}_+(I)$ such that $f(x) \geq x$ for all $x \in [p_0, q_0]$, and $Fix(f) \cap [p_0, q_0] = \{\frac{p+\frac{1}{2}}{2}, \frac{\frac{1}{2}+q}{2}\}$.

If $g \in \Gamma$ is sufficiently close to f (such an element g exists by the denseness of Γ) then for sufficiently big m, we have $W(g^m, h_{i_1}, \ldots, h_{i_r})(x) \neq x$ for all $x \in [p_0, q_0]$ where W is a word of width and height at most two. Contradiction. $\square$

4. Dense $\Rightarrow$ Infinite Girth

In this section we prove Theorem 1.2

Let Γ be a finitely generated dense subgroup of $\mathrm{Homeo}_+(I)$, m be a positive integer and $\{\gamma_1, \dots, \gamma_s\}$ be a finite set of generators of Γ. We will find $\eta \in \Gamma$ such that the generating set $\{\eta, \eta^m\gamma_1\eta^m, \dots, \eta^m\gamma_s\eta^m\}$ has no relation of length less than m.

Let F_1 be a free group formally generated by letters $\gamma, \gamma_1, \dots, \gamma_s$. (by abusing the notation, we treat the elements $\gamma_1, \dots, \gamma_s$ of Γ also as the elements of F_1.) Let also $a_0 = \gamma, a_1 = \gamma^m\gamma_1\gamma^m, \dots, a_s = \gamma^m\gamma_s\gamma^m$, and F_2 be free group formally generated by $a_0, a_1, \dots, a_s$. (so both F_1 and F_2 are free groups of rank $s+1$.)

Let $W_1, \dots, W_N$ be all reduced words in the free group F_2 of length at most m. These words can be written as reduced words $V_1, \dots, V_N$ in the free group F_1 where each word has length at most $m(2m+1)$.

Let $A = \{\gamma_1, \gamma_1^{-1} \dots, \gamma_s, \gamma_s^{-1}\}$. We will view A as a symmetrized generating set of the group Γ, and also as a finite subset of F_1. We will build disjoint finite subsets $S^{(0)}, S^{(1)}, \dots, S^{(N)}$ of $(0,1)$ and define an increasing map $f : \bigsqcup_{i=0}^{N} S^{(i)} \to (0,1)$ (i.e. if $x, y \in \bigsqcup_{i=0}^{N} S^{(i)}$ and $x < y$ then $f(x) < f(y)$) inductively as follows:

First, we let $S^{(0)} = \{\frac{1}{2}\}$, and $f(\frac{1}{2}) \notin \bigsqcup_{g\in A} g(\dfrac{1}{2})$.

Suppose now the subsets $S^{(0)}, S^{(1)}, \dots, S^{(k-1)}$ are chosen and the map f is defined on $\bigsqcup_{i=0}^{k-1} S^{(i)}$. We will describe how to define $S^{(k)}$ and extend the map f to $\bigsqcup_{i=0}^{k} S^{(i)}$.

Assume that V_k has length n as a reduced word in the free group F_1, and let $V_k = c_n \dots c_2c_1$ where $c_i \in \{\gamma, \gamma^{-1}, \gamma_1, \gamma_1^{-1}, \dots, \gamma_s, \gamma_s^{-1}\}$, and $U_i = c_i \dots c_2c_1, 1 \leq i \leq n$, (so $U_1, \dots, U_n$ are suffixes of V_k where the reduced word U_i has length $i, 1 \leq i \leq n$). We define the set $S^{(k)} = \{x_0, \dots, x_n\}$ itself and the map f on it [i.e. the sequence $f(x_0), \dots, f(x_n))$] inductively as follows:

We let x_0 be any point in $(0,1)$ such that

$$x_0 \notin \bigsqcup_{g\in A} g\Big(\bigsqcup_{1\leq i\leq k-1} S^{(i)}\Big) \cup f\Big(\bigsqcup_{1\leq i\leq k-1} S^{(i)}\Big)$$

Then we define $f(x_0) = y_0$ such that for all $x \in \bigsqcup_{1 \le i \le k-1} S^{(i)}$, we have $f(x) < y_0$ iff $x < x_0$ (so we extend the domain of f such that it stays being an increasing map).

Now assume that $x_1, \dots, x_r$ and $f(x_1), \dots, f(x_r)$ are defined.

We consider two cases:

Case 1. U_{r+1} starts with $\gamma^{\pm 1}$.

Then we choose x_{r+1} to be any point not in

$$\bigsqcup_{g \in A} g(\bigsqcup_{1 \le i \le k-1} S^{(i)}) \cup f(\bigsqcup_{1 \le i \le k-1} S^{(i)}) \sqcup \{x_0, \dots, x_r\} \cup \{f(x_0), \dots, f(x_r)\}$$

and let $f(x_{r+1}) = y_{r+1}$ where for all $x \in \sqcup_{1 \le i \le k-1} S_i \sqcup \{x_0, \dots, x_r\}$, we have $f(x) < y_{r+1}$ iff $x < x_{r+1}$.

Case 2. If U_{r+1} starts with some $g \in A$.

Then we let $x_{r+1} = g(x_r)$ and define $f(x_{r+1}) = y_{r+1}$ where for all $x \in \sqcup_{1 \le i \le k-1} S_i \sqcup \{x_0, \dots, x_r\}$, we have $f(x) < y_{r+1}$ iff $x < x_{r+1}$.

Thus we have constructed finite sets

$$S^{(1)} = \{x_0^{(1)}, x_1^{(1)}, \dots, x_{l_1}^{(1)}\}, \dots, S^{(N)} = \{x_0^{(N)}, x_1^{(N)}, \dots, x_{l_N}^{(N)}\}$$

corresponding to the words $V_1, \dots, V_N$ respectively and a map $f : \bigsqcup_{i=1}^{N} S^{(i)} \to (0,1)$ such the following conditions hold:

(i) $S^{(i)}$ consists of $(l_i + 1)$ points where l_i is the length of V_i as a reduced word in F_1;

(ii) $S^{(1)}, \dots, S^{(N)}$ are mutually disjoint;

(iii) for all $1 \le i \le N$, if $V_i = d_{l_i} \dots d_2 d_1$ where $d_j \in \{\gamma, \gamma^{-1}, \gamma_1, \gamma_1^{-1}, \dots, \gamma_s, \gamma_s^{-1}\}, 1 \le j \le l_i$, then $c_j \dots c_1(x_0^{(i)}) = x_j^{(i)}, 1 \le j \le l_i$ where

$$c_j = \begin{cases} d_i & \text{if } d_i \in \{\gamma_1, \gamma_1^{-1}, \dots, \gamma_s, \gamma_s^{-1}\} \\ f & \text{if } d_i = \gamma \\ f^{-1} & \text{if } d_i = \gamma^{-1} \end{cases}$$

(iv) $f : \bigsqcup_{i=0}^{N} S^{(i)} \to (0,1)$ is an increasing function.

By condition (iv), f can be extended to a homeomorphism $\eta \in \mathrm{Homeo}_+(I)$.

We claim that there is no relation of length less than $m+1$ among $\eta, \gamma_1, \dots, \gamma_s$. Indeed, let W be a reduced word of length at most m in the alphabet $a_1, \dots, a_s$. Then W can be written as a reduced word

$V_i = V_i(\gamma, \gamma_1, \ldots, \gamma_s)$ for some $i \in \{1, \ldots, N\}$. Then $V_i(\eta, \gamma_1, \ldots, \gamma_s)$ is a homeomorphism and $V_i(\eta, \gamma_1, \ldots, \gamma_s)(x_0^{(i)}) = x_{l_i}^{(i)} \neq x_0^{(i)}$.

Let $0 < \epsilon < \min_{1 \leq i \leq N} \frac{|x_{l_i}^{(i)} - x_1^{(i)}|}{2}$. The homeomorphism η is not necessarily in Γ, but let us recall that Γ is dense in $\mathrm{Homeo}_+(I)$. Then, if $\xi \in \Gamma$ is sufficiently close to η we will have $V_i(\xi, \gamma_1, \ldots, \gamma_s)(x_1^{(i)}) \in (x_{l_i}^{(i)} - \epsilon, x_{l_i}^{(i)} + \epsilon)$ for all $i \in \{1, \ldots, N\}$. Hence $V_i(\xi, \gamma_1, \ldots, \gamma_s) \neq 1 \in \Gamma$, for all $i \in \{1, \ldots, N\}$. Then there is no relation of length less than m among the elements of the generating set $\{\xi, \xi^m\gamma_1\xi^m, \ldots, \xi^m\gamma_s\xi^m\}$. Thus $girth(\Gamma) \geq m$. Since m is arbitrary, we conclude that $girth(\Gamma) = \infty$. $\square$

R e f e r e n c e s :

[A1] Akhmedov, A. On free discrete subgroups of Diff(I). *Algebraic and Geometric Topology*, **vol.4**, (2010) 2409-2418.

[A2] Akhmedov, A. A weak Zassenhaus Lemma for discrete subgroups of Diff(I). *Algebraic and Geometric Topology,* **14**, no.1, (2014) 539-550. http://arxiv.org/pdf/1211.1086.pdf

[A3] Akhmedov, A. Questions and remarks on discrete and dense subgroups of Diff(I). *Journal of Topology and Analysis*, **vol. 6**, no. 4, (2014), 557-571 http://arxiv.org/abs/1311.6782

[A4] Akhmedov, A. On the girth of finitely generated groups. *Journal of Algebra*, **268**, 2003, no.1, 198-208.

[A5] Akhmedov, A. The girth of groups satisfying Tits Alternative. *Journal of Algebra*, **287**, 2005, no.2. 275-282.

[A6] Akhmedov, A Girth alternative for subgroups of $\mathrm{PL}_+(I)$. http://arxiv.org/abs/1105.4908

[AST] Akhmedov, A., Stein, M., Taback, J. Free limits of Thompson's group F. Geometriae Dedicata, **vol. 11** (2011) no.1, 163-176.

[BE] Bartholdi, L., Erschler, A. Ordering the space of finitely generated groups http://arxiv.org/abs/1301.4669

[Be] Bergman, M. G. Right orderable groups that are not locally indicable. *Pacific Journal of Mathematics.* **147** Number 2 (1991), 243-248.

[BG1] Breuillard, E., Gelander, T. On dense free subgroups of Lie groups. *Journal of Algebra*, **vol.261**, no.2, 448-467.

[BG2] Breuillard, E., Gelander, T. A topological Tits alternative, *Annals of Mathematics* **166** (2007), no.2, 427-474.

[Br] Brin, M.: Free group of rank two is a limit of Thompson's group F. Groups Geometry Dynamics, **4** (2010) no.3, 433-454.

[Ch] Chou, C. Elementary amenable subgroups, *Illinois Journal of Mathematics*, **vol. 24**, issue 3 (1980), 396-407.

[Co] Cornulier, Yves De. Dense subgroups with Property (T) in Lie groups. *Comment. Math. Helv.* **83(1)**, 55-65, 2008.

[G] Ghys, E. Groups acting on the circle. *Enseign. Math. (2)* 47(3-4):329-407, 2001.

[M] Margulis, G. Some remarks on invariant means, *Monatshefte für Mathematik* **90** (3): 233235,

[Nak1] Nakamura, K. Ph.D Thesis, University of California Davis, 2008.

[Nak2] Nakamura, K. The Girth Alternative for Mapping Class Groups. http://arxiv.org, arXiv:1105.5422.

[Nav1] Navas, A. A finitely generated locally indicable group with no faithful action by C^1 diffeomorphisms of the interval. *Geometry and Topology*, **14** (2010) 573-584.

[Nav2] Navas, A. Groups of Circle Diffeomorphisms. Chicago Lectures in Mathematics, 2011. *http://arxiv.org/pdf/0607481*

[Ra] Raghunathan, M.S. *Discrete subgroups of semi-simple Lie groups.* Springer-Verlag, New York 1972. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68.

[Ro] Rosenblatt, J. Uniqueness of Invariant Means for Measure-preserving Transformations, *Transactions of AMS*, **265** (1981) 623-636.

[S] Sullivan, D For $n > 3$ there is only one finitely additive rotationally invariant measure on the n-sphere on all Lebesgue measurable sets, *Bulletin of the American Mathematical Society*, **4** (1): 121123

[Wi1] Winkelmann, J. Generic subgroups of Lie groups. *Topology*, **41**, (2002), 163-181.

[Wi2] Winkelmann, J. Dense random finitely generated subgroups of Lie groups. http://arxiv.org/abs/math/0309129

[Y] Yamagata, S. *The girth of convergence groups and mapping class groups.* Osaka Jounral of Mathematics, **volume 48**, no.1, 2011.

Azer Akhmedov, Department of Mathematics, North Dakota State University, Fargo, ND, 58108, USA
E-mail address: azer.akhmedov@ndsu.edu